Integral based Curvature Estimators in Digital Geometry

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Table of content

1. Introduction
2. Integral Invariant Theory
3. Integral based curvature estimator in digital space
4. Experimental evaluation
5. Conclusion & Future work
<table>
<thead>
<tr>
<th>Plan</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>Integral Invariant Theory</td>
</tr>
<tr>
<td>3</td>
<td>Integral based curvature estimator in digital space</td>
</tr>
<tr>
<td>4</td>
<td>Experimental evaluation</td>
</tr>
<tr>
<td>5</td>
<td>Conclusion &amp; Future work</td>
</tr>
</tbody>
</table>
Context

Differential quantities...
- for shape analysis, shape matching, ...
- for mathematical modeling of deformable objects (DIGITALSNOW project)

How to make an estimator?
- Experimental analysis of approximation errors on shapes with known Euclidean values
- Formal proof of convergence
- Computational cost & timing

⇒ Multigrid convergence framework
Let us consider a family $X$ of smooth and compact subsets of $\mathbb{R}^d$. We denote shape $X$ as $X \in X$, and $D_h(X)$ the digitization of $X$ in a $d-$dimensional grid of resolution $h$. More precisely, we consider classical Gauss digitization defined as

$$D_h(X) \overset{def}{=} \left( \frac{1}{h} \cdot X \right) \cap \mathbb{Z}^d$$

where $\frac{1}{h} \cdot X$ is the uniform scaling of $X$ by factor $\frac{1}{h}$. Furthermore, the set $\partial X$ denotes the frontier of $X$ (i.e. its topological boundary). The $h$-boundary $\partial_h X$ is a $d - 1$-dimensional subset of $\mathbb{R}^d$, which is close to $\partial X$. 
Multigrid convergence for local geometric quantities

**Definition**

A local discrete geometric estimator $\hat{E}$ of some geometric quantity $E$ is **multigrid convergent** for the family $X$ if and only if, for any $X \in X$, there exists a grid step $h_X > 0$ such that the estimate $\hat{E}(D_h(X), \hat{x}, h)$ is defined for all $\hat{x} \in \partial_h X$ with $0 < h < h_X$, and for any $x \in \partial X$,

$$\forall \hat{x} \in \partial_h X \text{ with } \|\hat{x} - x\|_{\infty} \leq h, |\hat{E}(D_h(X), \hat{x}, h) - E(X, x)| \leq \tau_{X,x}(h),$$

where $\tau_{X,x} : \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ has null limit at 0. This function defines the **speed of convergence** of $\hat{E}$ toward $E$ at point $x$ of $X$. The convergence is **uniform** for $X$ when every $\tau_{X,x}$ is bounded from above by a function $\tau_X$ independent of $x \in \partial X$ with null limit at 0.
# Digital Curvature Estimators

## Experimentally convergent in 2D
- MDCA estimator [Roussillon, T. and Lachaud, J.O., 2011]
  
  *Uses the most centered maximal Digital Circular Arc (DCA) to estimate the radius of the osculating circle.*

## Theoretically & Experimentally convergent in 2D
- BC curvature estimator [Esbelin, H.A. and Malgouyres, R., 2009]
  
  *Convergence speed in $O(h^{\frac{4}{3}})$*

## Non convergent in 3D
- Curvature estimation for digital surfaces based convolutions [Fourey, S. and Malgouyres, R., 2008]
Main contribution

Digital curvature estimators:
- defined in both 2D and 3D
- easy to implement
- multigrid convergence is theoretically proved with an uniform convergence speed in $O(h^{\frac{1}{3}})$
- experimental validation of multigrid convergence
Plan

1. Introduction

2. Integral Invariant Theory

3. Integral based curvature estimator in digital space

4. Experimental evaluation

5. Conclusion & Future work
**Definition**

Given $X \in \mathbb{X}$ and a radius $r \in \mathbb{R}^{+*}$, the volumetric integral $V_r(x)$ at $x \in \partial X$ is given by

$$V_r(x) \overset{def}{=} \int_{B_r(x)} \chi(p) dp$$

where $B_r(x)$ is the Euclidean ball (kernel) with radius $r$ and center $x$ and $\chi(p)$ the characteristic function of $X$. In dimension 2, we simply denote $A_r(x)$ such quantity.
Curvature information with Integration

Lemma [Pottmann2009]

For a sufficiently smooth shape $X$ in $\mathbb{R}^2$ $x \in \partial X$, we have

$$A_r(x) = \frac{\pi}{2} r^2 - \frac{\kappa(X, x)}{3} r^3 + O(r^4)$$

where $\kappa(X, x)$ is the curvature of $\partial X$ at $x$.

For a sufficiently smooth shape $X$ in $\mathbb{R}^3$ and $x \in \partial X$, we have

$$V_r(x) = \frac{2\pi}{3} r^3 - \frac{\pi H(X, x)}{4} r^4 + O(r^5)$$

where $H(X, x)$ is the mean curvature of $\partial X$ at $x$.

Local estimators $\tilde{\kappa}_r(x)$ and $\tilde{H}_r(x)$

$$\tilde{\kappa}_r(X, x) \overset{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}, \quad \tilde{H}_r(X, x) \overset{def}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$$

Then:

$$\tilde{\kappa}_r(X, x) = \kappa(X, x) + O(r), \quad \tilde{H}_r(X, x) = H(X, x) + O(r)$$
1. Introduction

2. Integral Invariant Theory

3. Integral based curvature estimator in digital space

4. Experimental evaluation

5. Conclusion & Future work
Proof process

\[ A_r(x) \rightarrow \overline{\text{Area}(D_h(B_r(x) \cap X), h)} \]

\[ \text{Convergence of } \hat{k}_r(D_h(X), x, h) \]

\[ + \text{ [Pottmann2009]} \]

\[ \text{Convergence of } \hat{k}_r(D_h(X), \hat{x}, h) \]

\[ \text{and } \hat{H}_r(D_h(X'), x, h) \]

\[ \text{and } \hat{H}_r(D_h(X'), \hat{x}, h) \]
Step 1a - Area/Volume estimation

Given digital shapes $Z \subset \mathbb{Z}^2$, the discrete area estimator by counting at step $h$ are defined:

$$\widehat{\text{Area}}(Z, h) \overset{\text{def}}{=} h^2 \text{Card}(Z)$$

If $Z = D_h(X)$:

$$\widehat{\text{Area}}(D_h(X), h) = \text{Area}(X) + O(h^\beta)$$

- $\beta = 1$ in general convex case [Gauss]
- $\beta = \frac{15}{11} - \epsilon$ when the shape boundary is $C^3$ with non-zero curvature [Huxley1990]

Given digital shapes $Z' \subset \mathbb{Z}^3$, the discrete area estimator by counting at step $h$ are defined:

$$\widehat{\text{Vol}}(Z', h) \overset{\text{def}}{=} h^3 \text{Card}(Z')$$

If $Z' = D_h(X')$:

$$\widehat{\text{Vol}}(D_h(X'), h) = \text{Vol}(X') + O(h^\gamma)$$

- $\gamma = 1$ in general convex case [Kratzel1988]
- $\gamma = \frac{243}{158}$ for smoother boundary [Guo2010]
Step 1b - Convergence of volumetric integral estimation

Lemma

\[ |\hat{\text{Area}}(D_h(B_r(x) \cap X), h) - A_r(x)| \leq K_1'(r) h^\beta \]

\[ \hat{\text{Area}}(D_h(B_r(x) \cap X), h) = r^2 \hat{\text{Area}}(D_{h/r}(B_1(\frac{1}{r} \cdot x) \cap \frac{1}{r} \cdot X), h/r) \]

\[ |\hat{\text{Area}}(D_h(B_r(x) \cap X), h) - A_r(x)| \leq K_1 h^\beta r^{2-\beta} \]

with \( 1 \leq \beta < 2 \).

Proof hints

- Rescale shapes \( Z \) to only a unit ball \( B_1 \)
- True for any point of \( \mathbb{R}^2 \)
Step 2a - Integral digital curvature estimators

Convergence of $\hat{\kappa}_r(\mathcal{D}_h(X), x, h)$ and $\hat{H}_r(\mathcal{D}_h(X'), x, h)$

Reminder:

$\tilde{\kappa}_r(X, x) \overset{\text{def}}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}$, \quad $\hat{H}_r(X, x) \overset{\text{def}}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$

Then, we can define:

Integral digital curvature estimator $\hat{\kappa}_r$ of a digital shape $Z$ at point $x \in \mathbb{R}^2$ and step $h$:

$\forall 0 < h < r, \hat{\kappa}_r(Z, x, h) \overset{\text{def}}{=} \frac{3\pi}{2r} - \frac{3\text{Area}(B_{r/h}(\frac{1}{h} \cdot x) \cap Z, h)}{r^3}.$

Integral digital curvature estimator $\hat{H}_r$ of a digital shape $Z'$ at point $x \in \mathbb{R}^3$ and step $h$:

$\forall 0 < h < r, \hat{H}_r(Z', x, h) \overset{\text{def}}{=} \frac{8}{3r} - \frac{4\text{Vol}(B_{r/h}(\frac{1}{h} \cdot x) \cap Z', h)}{\pi r^4}.$
Step 2b - Convergence when \( x \in \partial_{h}X \)

Rem: \( \tilde{\kappa}_{r}(X, x) \overset{\text{def}}{=} \frac{3\pi}{2r} - \frac{3A_{r}(x)}{r^{3}} \) and \( \tilde{\kappa}_{r}(X, x) = \kappa(X, x) + O(r) \)

\[
|\hat{\kappa}_{r}(D_{h}(X), x, h) - \kappa(X, x)| \leq O(r) + 3K_{1} \frac{h^{\beta}}{r^{1+\beta}}
\]

Let us set \( r = kh^{\alpha} \), then

\[
|\hat{\kappa}_{r}(D_{h}(X), x, h) - \kappa(X, x)| \leq K_{2} k^{\alpha} h^{\alpha} + \frac{3K_{1}}{k^{1+\beta}} h^{\beta-\alpha(1+\beta)}
\]

Theorem (Convergence of digital curvature estimator \( \hat{\kappa}_{r} \) along \( \partial X \))

Let \( X \) be some convex shape of \( \mathbb{R}^{2} \), with at least \( C^{2} \)-boundary and bounded curvature. Then \( \exists h_{0}, K_{1}, K_{2} \), such that

\[
\forall h < h_{0}, r = k_{m} h^{\alpha_{m}}, |\hat{\kappa}_{r}(D_{h}(X), x, h) - \kappa(X, x)| \leq K h^{\alpha_{m}},
\]

where \( \alpha_{m} = \frac{\beta}{2+\beta}, k_{m} = ((1 + \beta)K_{1}/K_{2})^{\frac{1}{2+\beta}}, K = K_{2}k_{m} + 3K_{1}/k_{m}^{1+\beta} \).

\( \alpha_{m} = \frac{15}{37} - \epsilon \approx 0.405 \) when the boundary of \( X \) is \( C^{3} \) without null curvature points,
\( \alpha_{m} = \frac{1}{3} \) otherwise.
Step 3a - Convergence of $\hat{x} \in \partial_h X$

$\hat{x}$ lies on the normal direction to $\partial X$ at $x$, at a distance

$\delta \overset{def}{=} \|x - \hat{x}\|_2 \overset{def}{=} h \alpha'$

$$|A_r(\hat{x}) - A_r(x)| = 2r\delta(1 + O(r^2) + O(\delta)) \quad \text{[Pottmann2009]}$$

$$|\text{Area}(D_h(B_r(\hat{x}) \cap X), h) - A_r(x)| \leq K_1 h^\beta r^{2-\beta} + 2r\delta(1 + O(r^2) + O(\delta))$$

$$|\kappa_r(D_h(X), \hat{x}, h) - \kappa(X, x)| \leq O(r) + 3K_1 \frac{h^\beta}{r^{1+\beta}} + \frac{6\delta}{r^2}(1 + O(r^2) + O(\delta))$$

Back-projection $\pi^X_h$ [Lachaud2006]

Let $\hat{x} \in \partial_h X$ and set $x_0 = \pi^X_h(\hat{x})$.

$$\|\hat{x} - x_0\|_{\infty} \leq \frac{\sqrt{2}}{2} h < h$$
Step 3b - Convergence when $\hat{x} \in \partial_h X$

Convergence of $\hat{\kappa}_T (D_h(X), \hat{x}, h)$ and $\hat{H}_T (D_h(X'), \hat{x}, h)$

Theorem (Uniform convergence of curvature estimator $\hat{\kappa}_T$ along $\partial_h X$)

Let $X$ be some convex shape of $\mathbb{R}^2$, with at least $C^3$-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0$, setting $r = kh^\alpha$, $\delta = O(h^{\alpha'})$ where $\alpha \geq 1$, we have

$$\forall x \in \partial X, \forall \hat{x} \in \partial_h X, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$

$$|\hat{\kappa}_T (D_h(X), \hat{x}, h) - \kappa(X, x)| \leq O(h^\alpha)$$

$$+ O(h^{\beta - \alpha(1+\beta)})$$

$$+ O(h^{\alpha' - 2\alpha}) + O(h^{\alpha'}) + O(h^{2\alpha' - 2\alpha})$$

Finding the best possible parameter $\alpha_m = \frac{\beta}{1+\beta}$ if $\alpha' \geq \frac{3\beta}{1+\beta}$, otherwise $\alpha_m = \frac{\alpha'}{3}$

[Gauss] $\Rightarrow \beta = 1$
[Lachaud2006] $\Rightarrow \alpha' = 1$ \Rightarrow $\alpha_m = \frac{1}{3}$ \Rightarrow $|\hat{\kappa}_T (D_h(X), \hat{x}, h) - \kappa(X, x)| \leq Kh^{\frac{1}{3}}$
In the same way, we have in 3D:

**Theorem (Uniform convergence of $\hat{H}_r$ along $\partial hX$)**

Let $X'$ be some convex shape of $\mathbb{R}^3$, with at least $C^2$-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0 \ \forall x \in \partial X', \forall \hat{x} \in \partial hX', \|\hat{x} - x\|_{\infty} \leq h$

$$\forall 0 < h < r, \hat{H}_r(\partial hX', \hat{x}, h) \stackrel{def}{=} \frac{8}{3r} - \frac{4\text{Vol}(B_{r/h}(\hat{x}) \cap \partial hX', h)}{\pi r^4}.$$ 

Setting $r = k' h^{\frac{1}{3}}$, we have

$$|\hat{H}_r(D_h(X'), \hat{x}, h) - H(X', x)| \leq K' h^{\frac{1}{3}}.$$
Gaussian Curvature on Digital Surface

Main idea

- Instead of computing the volume of \( Y = B_r(x) \cap X \), we compute its covariance matrix

\[
J(Y) \overset{def}{=} \int_Y (p - \overline{Y})(p - \overline{Y})^T \, dp = \int_Y pp^T \, dp - \text{Vol}(Y)\overline{Y}\overline{Y}^T,
\]

where \( \overline{Y} \) denotes the centroid of \( Y \).

- Principal curvatures \( k^1 \) and \( k^2 \) at \( x \) are related to eigenvalues of \( J(Y) \)

\[
\lambda_1 = \frac{2\pi}{15} R^5 - \frac{\pi}{48} (3\kappa^1(X, x) + \kappa^2(X, x)) R^6 + O(R^7)
\]

\[
\lambda_2 = \frac{2\pi}{15} R^5 - \frac{\pi}{48} (\kappa^1(X, x) + 3\kappa^2(X, x)) R^6 + O(R^7)
\]

\[
\lambda_3 = \frac{19\pi}{480} R^5 - \frac{9\pi}{512} (\kappa^1(X, x) + \kappa^2(X, x)) R^6 + O(R^7)
\]

⇒ From convergence of high order moment estimator and specific error propagation analysis, convergence proofs can be designed for \( \kappa^1 \) and \( \kappa^2 \)
1 Introduction

2 Integral Invariant Theory

3 Integral based curvature estimator in digital space

4 Experimental evaluation

5 Conclusion & Future work
Experimental Settings

- Family of Euclidean shapes (implicit, parametric) with exact curvature information
- Digitization process at resolution $h$
- Error metrics
  - Worst-case $l_\infty$ error: maximum of absolute difference value
    $$\max_{\hat{x} \in \partial h, x \in \partial X} \left| \hat{\kappa}_r (D_h(X), \hat{x}, h) - \kappa(X, x) \right|$$
  - Quadratic $l_2$ error
Validation of $\alpha$ parameter

Convolution kernel radius

\[ r = k h^\alpha \]

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Observed convergence speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>$O(h^{0.024})$</td>
</tr>
<tr>
<td>2/5</td>
<td>$O(h^{0.24})$</td>
</tr>
<tr>
<td>1/3</td>
<td>$O(h^{0.38})$</td>
</tr>
<tr>
<td>2/7</td>
<td>$O(h^{0.41})$</td>
</tr>
<tr>
<td>1/4</td>
<td>$O(h^{0.44})$</td>
</tr>
</tbody>
</table>
Validation of $\alpha$ parameter

Convolution kernel radius

$r = kh^{\alpha}$

![Graph showing the relationship between $L_{\infty}$ error and $h$ for different $\alpha$ values. The graph illustrates how the error changes with varying kernel radius $r$. The $x$-axis represents $h$ ranging from 0.0001 to 1, and the $y$-axis represents $L_{\infty}$ error ranging from 0.1 to 10. Different lines correspond to different $\alpha$ values such as $\alpha=1/2$, $\alpha=2/5$, $\alpha=1/3$, $\alpha=2/7$, and $\alpha=1/4$, each showing a distinct pattern of error variation.]
Validation of $\alpha$ parameter

Convolution kernel radius

$r = kh^\alpha$

Implicit surface is $x^2 + y^2 + z^2 - 25 = 0$
Comparison

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Observed convergence speed</th>
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<tbody>
<tr>
<td>II</td>
<td>$O(h^{0.38})$</td>
</tr>
<tr>
<td>MDCA</td>
<td>$O(h^{0.42})$</td>
</tr>
<tr>
<td>BC</td>
<td>$O(h^{0.154})$</td>
</tr>
</tbody>
</table>

Kanungo noise

noise parameter: $0.5 \ (\in [0, 1])$
3D curvature estimation
Optimizations with convolution

- Open-source C++ library
- Geometry structures, algorithm & tools for digital data
- http://libdgtal.org

Optimization with displacement masks

Complexity:
- without optimization: $O((r/h)^d)$
- with optimization: $O((r/h)^{d-1})$
Plan

1. Introduction
2. Integral Invariant Theory
3. Integral based curvature estimator in digital space
4. Experimental evaluation
5. Conclusion & Future work
Conclusion & Future work

- Integral Invariant is perfect for digital geometry
- A unique estimator for both 2D and 3D
- Easy to implement
- Fast computation with masks
- Convergent with a least a uniform convergence speed in $O(h^{\frac{1}{3}})$
- Needs a parameter ($r$ for the kernel radius), as BC .. but we have better results

For an Ellipse, and the same $L_\infty$ error between BC and our estimator ($0.0461726$), we have:

<table>
<thead>
<tr>
<th></th>
<th>BC</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.00302755</td>
<td>0.0301974</td>
</tr>
<tr>
<td>time (in ms.)</td>
<td>421945</td>
<td>350</td>
</tr>
<tr>
<td>mask size</td>
<td>91336</td>
<td>8349</td>
</tr>
<tr>
<td>mask size (optim.)</td>
<td>8*145</td>
<td>8*145</td>
</tr>
</tbody>
</table>

Future work

- Theoretical demonstration of Gaussian curvature convergence
- Comportment with noised data
- Scale-Space analysis
Choice of radius
For more information ($l_2$ graphs, high-res images, scripts, etc.):

http://liris.cnrs.fr/jeremy.levallois/Papers/DGCI2013/