On the Non-Additive Sets of Uniqueness in a Finite Grid

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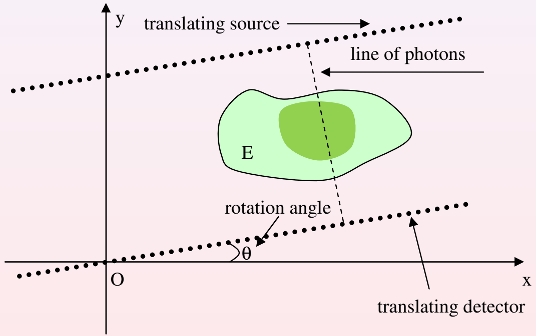
DGCI 2013

Inverse Problem in Discrete Tomography

Inverse problem in Tomography: reconstruct a set *E* from X-ray data.

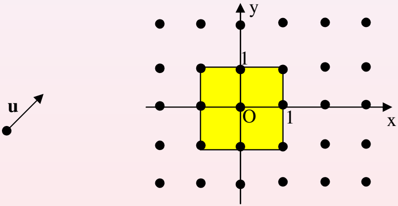


The notion of X-ray changes according to the different settings.

Discrete X-rays

In Discrete Tomography discrete X-rays are employed.

The discrete (parallel) X-ray of a set E in the direction u counts the number of point of E on each line parallel to u.



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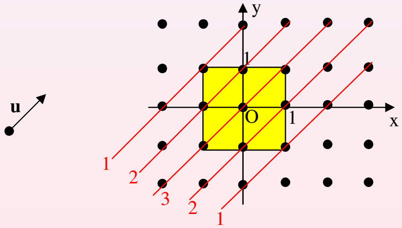
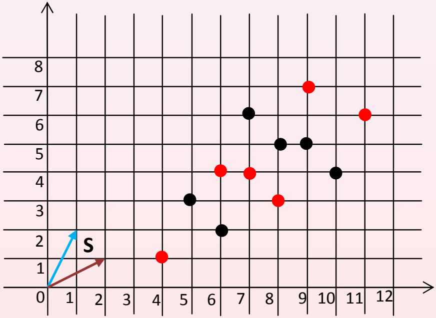


Problem. Can uniqueness of reconstruction be ensured by X-rays taken in suitable finite sets *S* of lattice directions?

Switching components (or bad-configurations) must be avoided.



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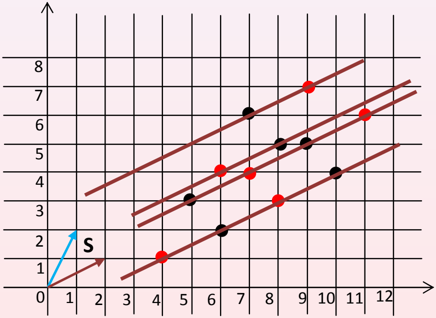

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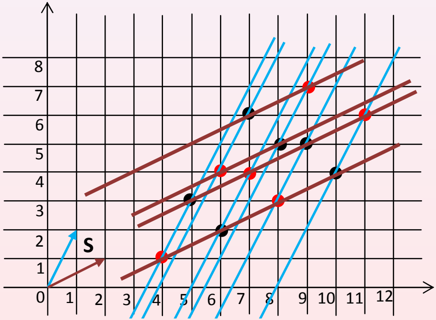

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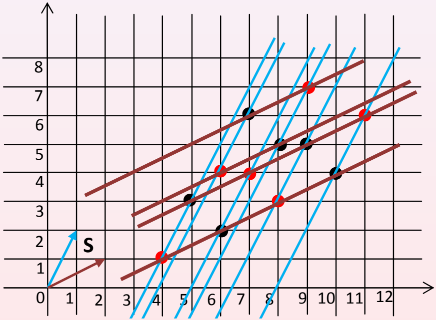


Characterization of switching components. [L.Hajdu-R.Tijdeman, 2001]

Uniqueness and Additivity

Additivity is often useful.

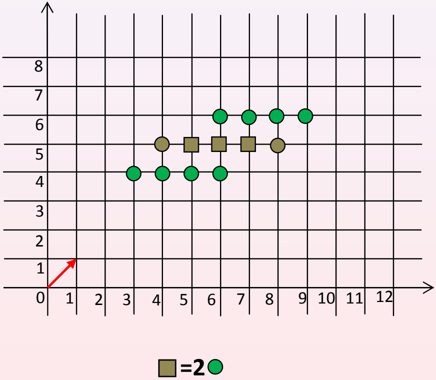
A finite set $E \subseteq \mathbb{Z}^2$ is *S-additive* if for each $u \in S$, there is a *ridge function* g_u, that is a function defined in $\mathbb{Z}^2$ which is constant on each lattice line parallel to u, such that

$$E = \left\{ p \in \mathbb{Z}^2 : \sum_{u \in S} g_u(p) > 0 \right\}.$$

A set E is additive if and only if it has no weakly bad configurations (switching components with multiple points).

[P. Fishburn et al.,1991]

Example



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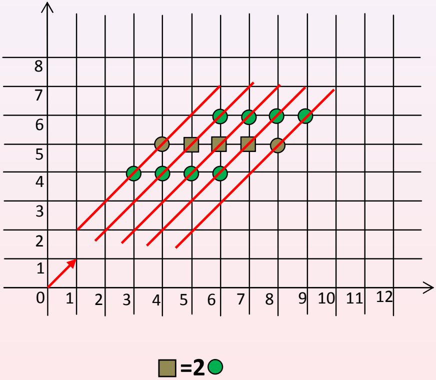


Uniqueness and Additivity

• $n = 2$, $S = \{(1,0),(0,1)\}$, F is S-unique $\Leftrightarrow$ F is additive.
[P. Fishburn et al.,1991]

• $n = 2$, $|S| > 2$, F is additive $\Rightarrow$ F is S-unique, but there exist non-additive sets of uniqueness. [P. Fishburn et al.,1997]

• Higher dimension generalizations.
-Additivity $\Rightarrow$ Uniqueness (not conversely) [P. Fishburn-L.Shepp, 1999]
-Pyramids [E.Vallejo, 1997,2002,2005,2007]; [M.Santoyo, E.Vallejo, 2008], [P.D.,C.Peri,2012]

Conjecture **[P.Fishburn-L.Shepp, 1999]** For fixed S of cardinality larger than 2 the proportion of lattice sets E of uniqueness that are also non-additive approaches 1 as E gets large.

Uniqueness and Additivity

Problem. What about if sets confined in a fixed grid $\mathcal{A}$ are considered?

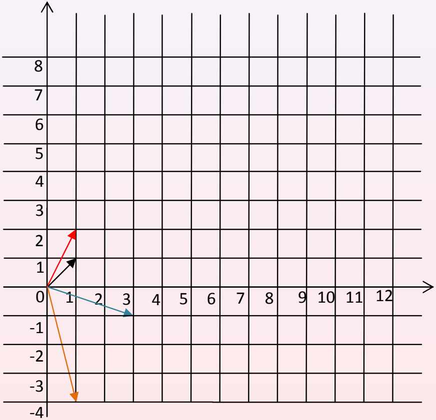
Sets S of *valid* directions for $\mathcal{A}$ must be employed.

• $|S| < 4$ NO UNIQUENESS [L.Hajdu, 2005].
• $|S| = 4$.
- Example of unique reconstruction for some particular S [L.Hajdu, 2005].
- Generalization to whole families of four directions [S.Brunetti, P.D., C.Peri, 2011].
- A necessary and sufficient condition for S-uniqueness in $\mathcal{A}$ [S.Brunetti, P.D., C.Peri, 2012].

Problem. Find an algorithm that constructs non-additive sets of uniqueness.

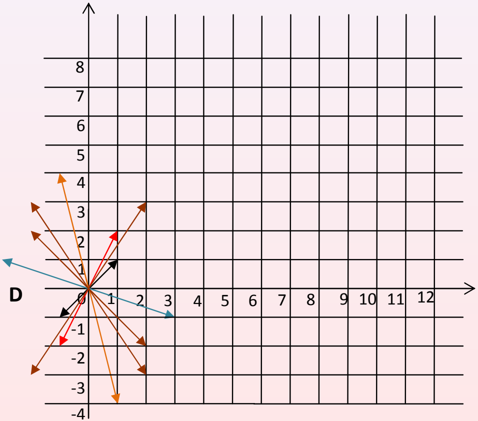
Constructing non additive sets of uniqueness

Take $S = \{u_r = (a_r, b_r),\ r = 1, 2, 3, 4\}$, $u_4 = u_1 + u_2 \pm u_3$.

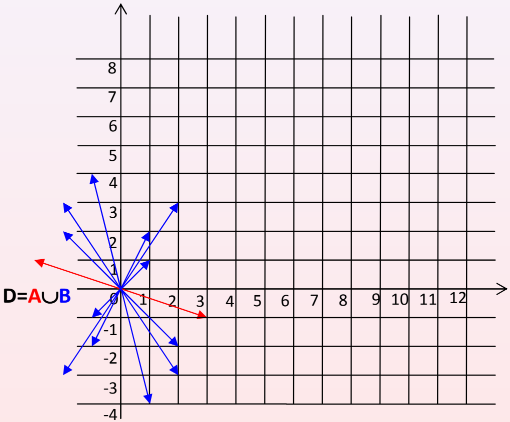


$S = \{(1, 1), (1, 2), (1, -4), (3, -1)\}$

Take $\widehat{S} = \left\{ \pm (v - u_4) : v \in \{u_1, u_2, u_4 - u_1 - u_2\} \right\}$ and form
$D = \pm S \cup \widehat{S} = =$
$\{\pm(1,1), \pm(1,2), \pm(1,-4), \pm(3,-1), \pm(-2,2), \pm(-2,3), \pm(-2,-3)\}$



$A = \{\pm(3, -1)\}$,
$B = \{\pm(1, 1), \pm(1, 2), \pm(1, -4), \pm(-2, 2), \pm(-2, 3), \pm(-2, -3)\}$



$$A = \{(a, b) \in D : \ |a| > |b|\}, \quad B = \{(a, b) \in D : \ |b| \geq |a|\}.$$

Constructing non additive sets of uniqueness

Select a grid $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \,:\, 0 \le i < m,\ 0 \le j < n\}$ where

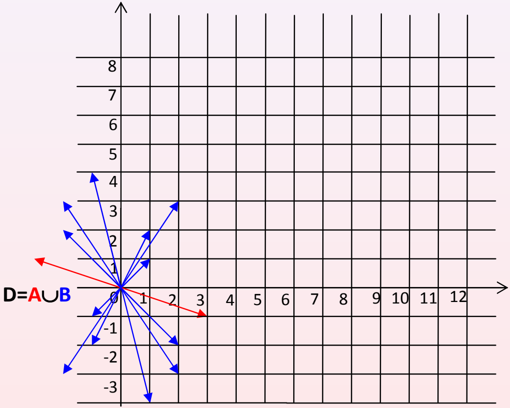
$$n \le \min\left\{\min_{|a|} A + \sum_{r=1}^{4} |b_r|, \min_{|b|} B + \sum_{r=1}^{4} |b_r|\right\},$$

$$m > n + \sum_{r=1}^{4} a_r - \sum_{r=1}^{4} |b_r|.$$

[S.Brunetti, P.D., C.Peri, 2012].

In our example $n \le 9$ and $m > n$.

For instance $n = 9, m = 12$ is allowed.



Compute $F_S(x,y) = \prod_{(a,b)\in S} f_{(a,b)}(x,y)$, where

$$f_{(a,b)}(x,y) = \begin{cases} x^a y^b - 1, & \text{if } a > 0, b > 0 \\ x^a - y^{-b}, & \text{if } a > 0, b < 0 \\ x - 1, & \text{if } a = 1, b = 0 \\ y - 1, & \text{if } a = 0, b = 1. \end{cases}$$

Therefore, with $S = \{(1,1),(1,2),(1,-4),(3,-1)\}$

$$F_S(x,y) = (xy-1)(xy^2-1)(x-y^4)(x^3-y).$$

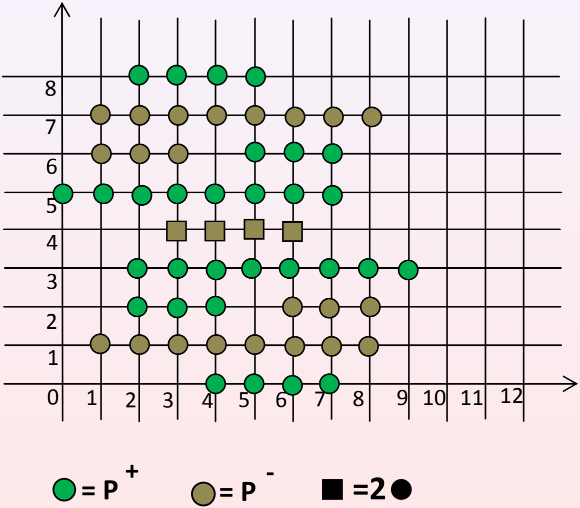
Constructing non additive sets of uniqueness

Consider a polynomial $P(x,y) = H(x,y)F_S(x,y)$ such that
$\deg_x P(x,y) < m, \quad \deg_y P(x,y) < n$.
Take for instance $P(x,y) = (x^3 + x^2 + x + 1)F_S(x,y)$
Then we get

$$
\begin{aligned}
&P(x,y) = \\
&= x^9y^3 - x^8y^7 + x^8y^3 - x^8y^2 - x^8y - x^7y^7 + x^7y^6 + x^7y^5 + \\
&+ x^7y^3 - x^7y^2 - x^7y + x^7 - x^6y^7 + x^6y^6 + x^6y^5 - 2x^6y^4 + \\
&+ x^6y^3 - x^6y^2 - x^6y + x^6 + x^5y^8 - x^5y^7 + x^5y^6 + x^5y^5 - \\
&- 2x^5y^4 + x^5y^3 - x^5y + x^5 + x^4y^8 - x^4y^7 + x^4y^5 - 2x^4y^4 + \\
&+ x^4y^3 + x^4y^2 - x^4y + x^4 + x^3y^8 - x^3y^7 - x^3y^6 + x^3y^5 - \\
&- 2x^3y^4 + x^3y^3 + x^3y^2 - x^3y + x^2y^8 - x^2y^7 - x^2y^6 + x^2y^5 + \\
&+ x^2y^3 + x^2y^2 - x^2y - xy^7 - xy^6 + xy^5 - xy + y^5
\end{aligned}
$$

Each set $E \subset \mathcal{A}$ such that $P^+ \subset E$ and $P^- \cap E = \emptyset$ (or $P^- \subset E$ and $P^+ \cap E = \emptyset$) is a non-additive set of uniqueness.

Constructing non additive sets of uniqueness



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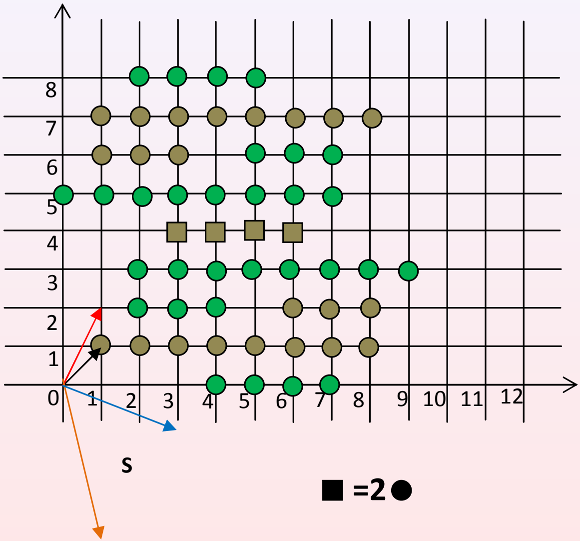

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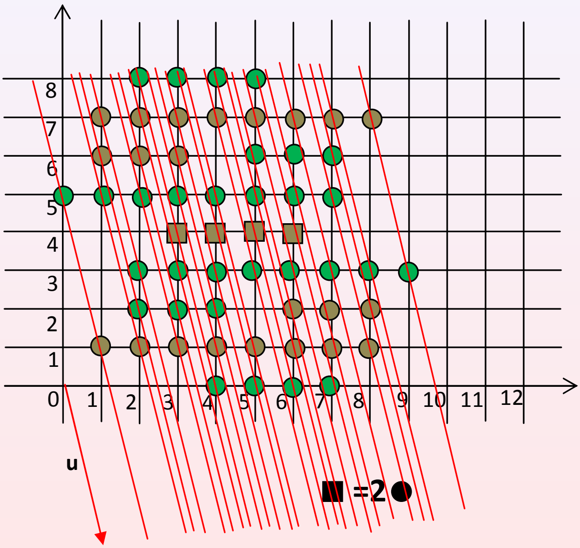

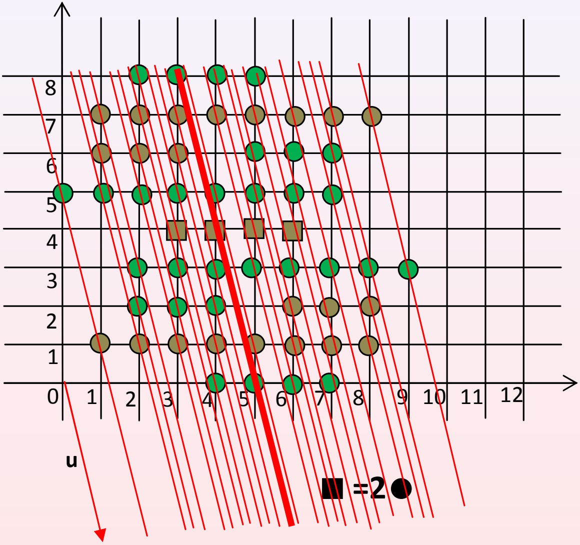

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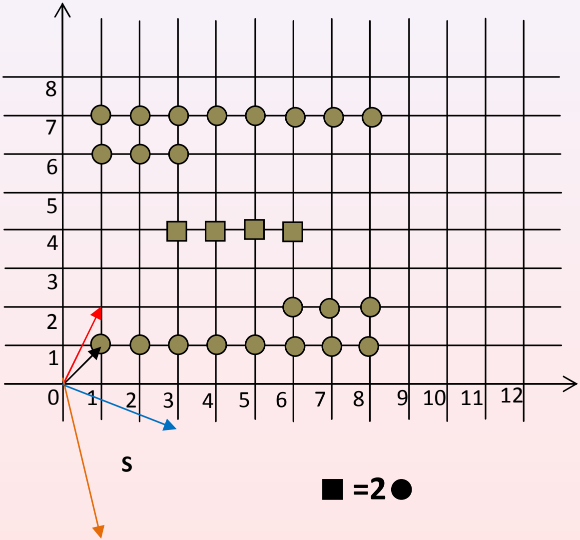
Constructing non additive sets of uniqueness

Points with a same color form non-additive sets of uniqueness.



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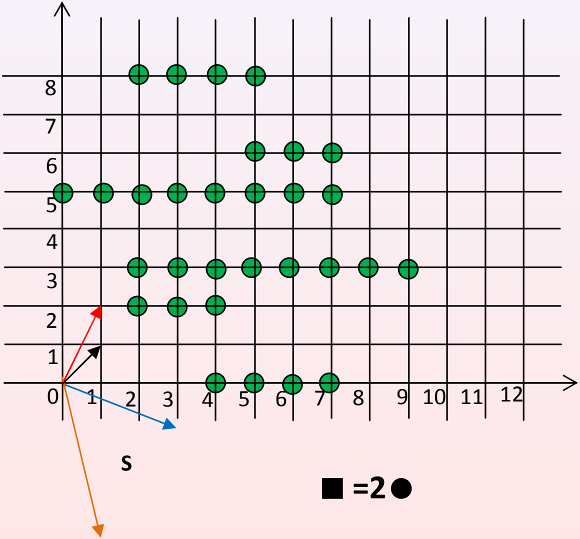

Finding additive sets

In the case when $\{(1,0),(0,1)\} \subset S$, we can get a complete classification of bounded sets.

- If S is not valid for $\mathcal{A}$, then all bounded sets are additive.
- If S is valid for $\mathcal{A}$, then:
  - every bounded set E such that ${F_S}^+ \subseteq E \ \wedge \ E \cap {F_S}^- = \emptyset$, or ${F_S}^- \subseteq E \ \wedge \ E \cap {F_S}^+ = \emptyset$ is a non-additive set of uniqueness;
  - all the other bounded sets are additive.

The rate of non-additivity in a grid

The ratio between bounded non-additive sets of uniqueness and additive sets can be computed.

$$R = \frac{2^{|\mathcal{A}\setminus F_S|+1}}{2^{|\mathcal{A}|} - 2^{|\mathcal{A}\setminus F_S|+1}} = \frac{2}{2^{|F_S|} - 2} = (2^{14} - 1)^{-1},$$

approximately equal to 6.1×10^{-5}.

This partially answer in the negative the conjecture of [P.Fishburn-L.Shepp, 1999] that for some set of X-ray directions of cardinality larger than 2 the rate approaches 1 for increasing size of the sets

To resume

- Uniqueness of reconstruction in a finite grid.
- Explicit procedure to get non-additive sets of uniqueness.
- Very low R when the coordinate directions belong to S, independently of the size (against Fishburn-Shepp conjecture).
- What about R for different sets S of uniqueness? We are investigating such an issue