Walking in the Farey Fan to Compute the Characteristics of a Discrete Straight Line Subsegment

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Outline

1. Setting the problem
2. Properties of the Farey Fan
3. Fast walk in the Farey Fan
4. Experiments

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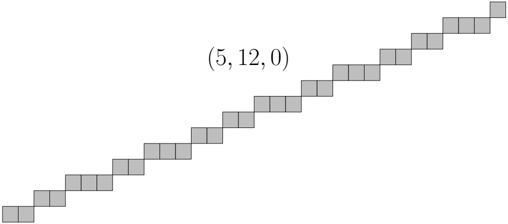
DSS and minimal characteristics

Digital Straight **Line** [Reveilles, 1991]

A DSL of integer characteristics (a, b, μ) is the infinite set of digital points (x, y) such that $0 \leq ax - by + \mu < \max(a, b)$.

Digital Straight **Segment**

A DSS is a finite 8-connected part of a DSL.



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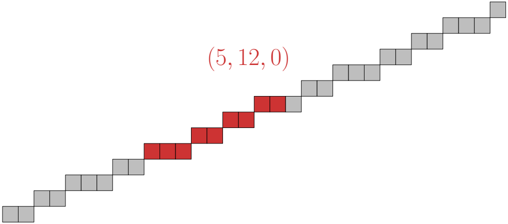

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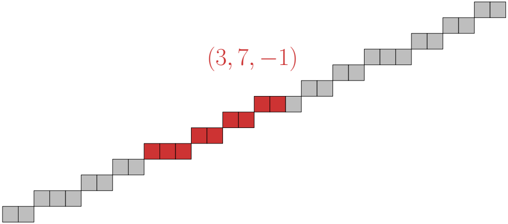

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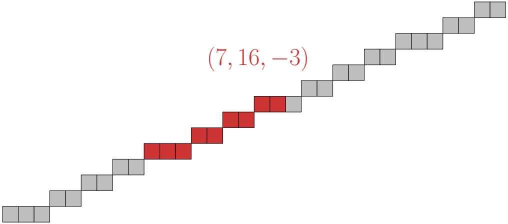

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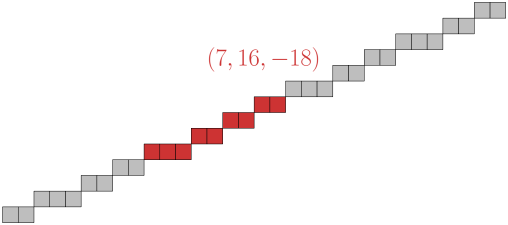

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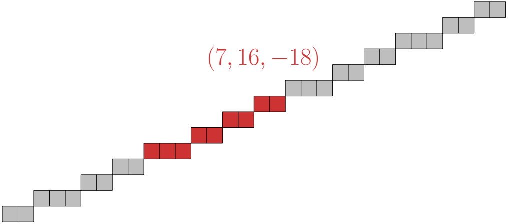

Minimal charac. of a DSS [Sivignon et al. 2004] [Said, Lachaud 2011]

Among all the DSL of charac. (a_i, b_i, μ_i) containing the DSS, take the one with **minimal** b_i.

DSL subsegment

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minimal charac. of a DSS = result of classical recog. algorithms
$\Longrightarrow$ **linear complexity** [Debled, Reveilles 1995][Troesh 1993][Dorst, Smeulders 1991][...]

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Do better when a DSL container is known ?

- Two algorithms by [Said & Lachaud 2011] : SmartDSS and ReversedSmartDSS ;
- Based on the continued fraction of the slope + recursive computation of the pattern of a DSS ;
- Complexity : $\mathcal{O}(\sum_{i=0}^{k} u_i)$ and
  $\mathcal{O}($*depth of input slope - depth of output slope*$)$

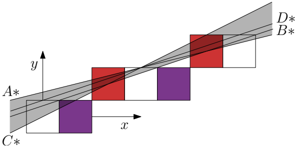
Minimal charac. of a DSS [Sivignon et al. 2004] [Said, Lachaud 2011]

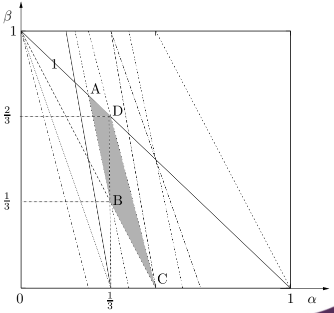
Among **all the DSL** of charac. (a_i, b_i, μ_i) containing the DSS, take the one with **minimal** b_i.

Preimage of a DSS

Set of all the DSL containing a DSS

$\mathcal{P}(S) = \{(\alpha, \beta), |\alpha| \leq 1 \mid \forall (x, y) \in S, 0 \leq \alpha x - y + \beta < 1\}$





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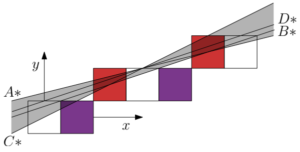
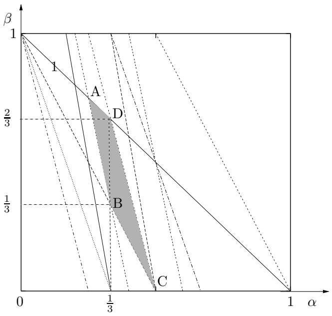




Coordinates of vertex $B \Leftrightarrow$ minimal charac. of the DSS.
B is the **characteristic point**.
The characteristic point is on a **lower edge**.

Farey fan

Ray

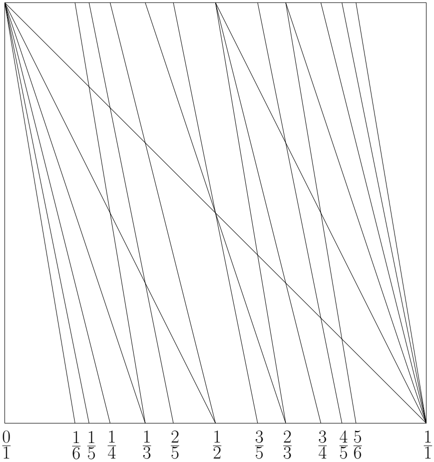
$R(x, y) = \{(\alpha, \beta) | \beta = -x\alpha + y\}$

x is the *slope* of the ray.

Farey fan of order n

Arrangement of all the rays $R(x, y)$:

- $x, y \in \mathbb{Z}$
- $0 \leq y \leq x \leq n$
- $\alpha, \beta \in [0, 1]$



$\mathcal{F}_6$

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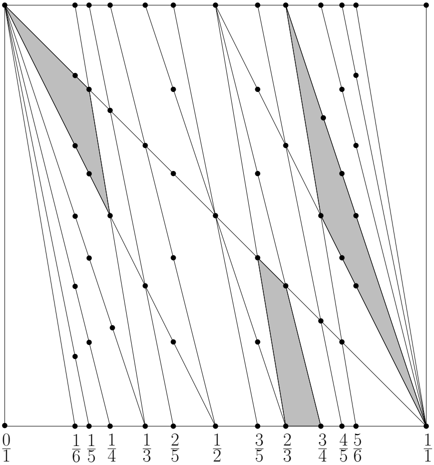


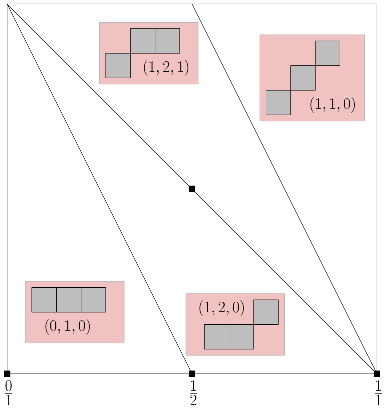
$\mathcal{F}_6$

- **facet** = cell of dimension 2
- **point** = point on a ray with denom. of abs. lower than n.

Preimage as a cell of a Farey Fan

Key property [McIlroy 1985]

Bijection between the facets of $\mathcal{F}_n$ and the DSSs of length n ($n+1$ pixels).



[McIlroy, M.D. : A Note on Discrete Representation of Lines. AT&T Technical Journal 64(2), 481-490 (1985)]

Rewriting the DSL subsegment problem

In the digital space

Given a DSL L of charac. (a, b, μ) and two points P and Q of L, compute the minimal charac. of the DSS $S = \{(x, y) \in L \mid x_P \leq x \leq x_Q\}$.

$\Downarrow$ (+ translation to set P as the origin)

In the Farey fan

Given a point $\Lambda(\frac{a}{b}, \frac{\mu}{b})$ and a Farey fan of order $n = x_Q - x_P$, find the characteristic point of the facet in which Λ lies.

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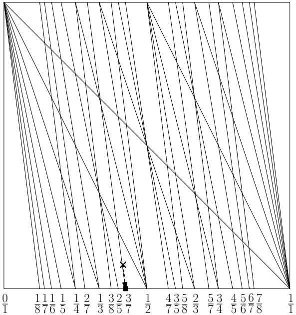

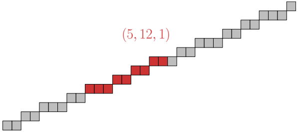

Point location algorithm

- point location in an arrangement of $\mathcal{O}(n^3)$ facets...
- take profit of arithmetical properties to do better [McIlroy 1985]

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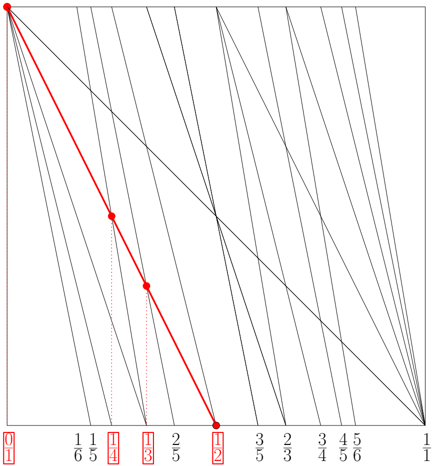
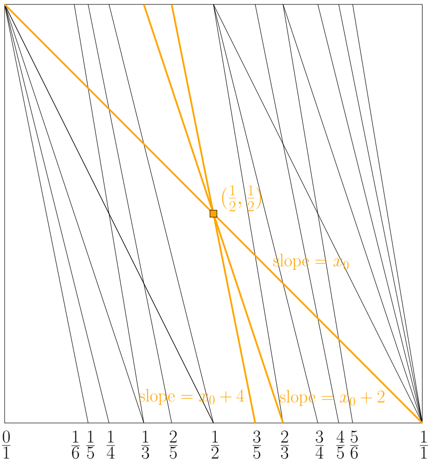
Farey fan properties and fast computations/tests



Abscissas of intersections of a ray of slope x with others are consecutive terms of a Farey series of order $\max(x, n - x)$.

$\Longrightarrow$ fast move from one intersection point to the other along a ray

Farey fan properties and fast computations/tests



The rays passing through $(\frac{p}{q}, \frac{r}{q})$ have a slope equal to $x_0 + kq$ with $x_0 \in [0, q[$, $k \in \mathbb{Z}$ and $x_0 + kq < n$.

$\Longrightarrow \mathcal{O}(1)$ test to decide if a ray is the one of smallest/greatest slope

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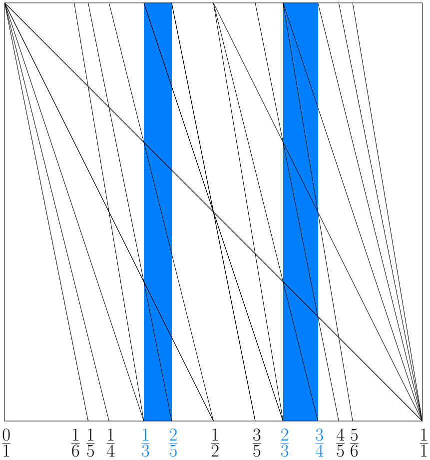

No intersection of rays between two consecutive fractions of $\mathcal{F}_n$
$\Longrightarrow$ ladder

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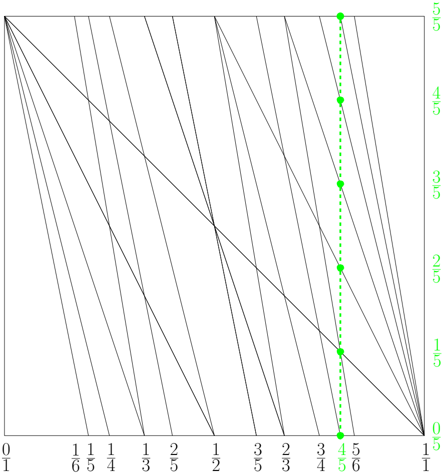

Intersection points between $\alpha = \frac{p}{q}$ and $\mathcal{F}_n$ have ordinates equal to $\frac{r}{q}$ where r takes all the values between 0 and q.

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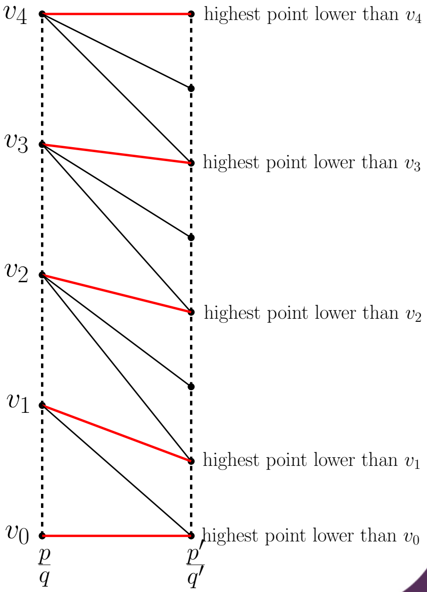
+

Intersection points between $\alpha = \frac{p}{q}$ and $\mathcal{F}_n$ have ordinates equal to $\frac{r}{q}$ where r takes all the values between 0 and q.

$\Longrightarrow \mathcal{O}(1)$ computation of the ray of smallest slope in a ladder



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A three steps algorithm

Algorithm [similar to McIlroy 1985]

1. find the ladder in which Λ lies
2. locate the highest ray that lies on or below $\Lambda \Rightarrow$ lower edge
3. walk along the ray(s) to find the charac. point

- algorithm not detailed in [McIlroy 1985]
- announced complexity $\mathcal{O}(\log^2 n)$

$\Rightarrow$ detailed algorithm with complexity lowered to $\mathcal{O}(\log n)$

Find the ladder

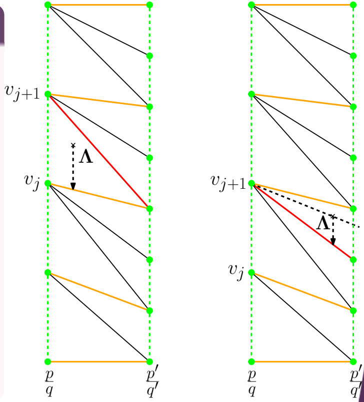
$\Lambda(\frac{a}{b}, \frac{r}{b})$

Find the two fractions with a denominator smaller than n closest to $\frac{a}{b}$ (greater and lower).

Best approximation of a number with a limited denominator
$\Longrightarrow$ Algorithm of [Charrier, Buzer 2009] in $\mathcal{O}(log(n))$

Dichotomic search

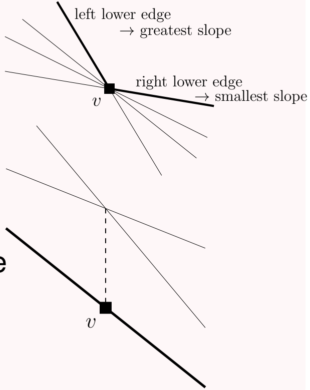
$R_i(x_i, y_i)$ = ray of smallest slope through v_i
Dichotomy on the R_i to compute j such that $\mathbf{\Lambda}$ is above R_j and below R_{j+1}
if $\mathbf{\Lambda}$ *is under the ray of greatest slope through* v_{j+1} **then**
 Return R_j
else
 Compute the slope x of $(v_{j+1}, \mathbf{\Lambda})$
 Compute $[x]$ as the value $x_{j+1} + kq$ nearest to and lower than x
 Return $R([x], \frac{(j+1)+p.[x]}{q})$



Characteristic point

$v(\frac{p_v}{q_v}, \frac{r_v}{q_v})$ characteristic point of a facet iff :

- either v is the intersection of the two lower edges
- or v is on the unique lower edge and there is more than one ray through the point $(\frac{p_v}{q_v}, \frac{r_v+1}{q_v})$



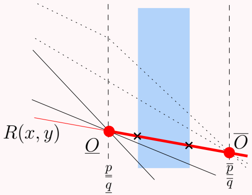
Find the characteristic point

Walk on the ray R to find facet vertices

Compute points $\underline{O}$ and $\overline{O}$ as the previous and next points with abscissa of denominator $\leq \max(x, n-x)$.

$\underline{O}$?

is R the ray of smallest slope ?



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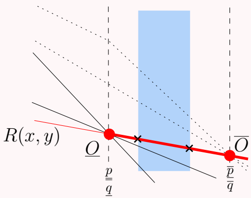

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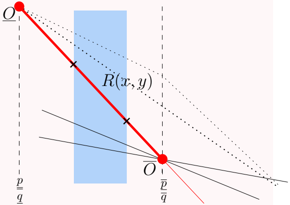

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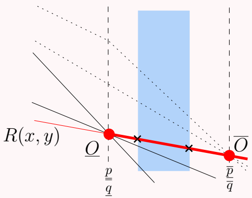

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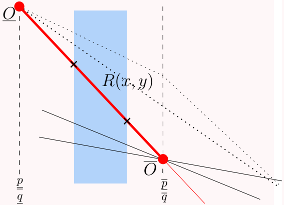
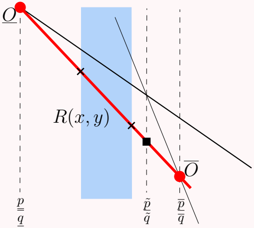


otherwise...

compute the mediant



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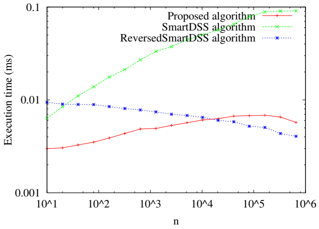
Comparison with other algorithms

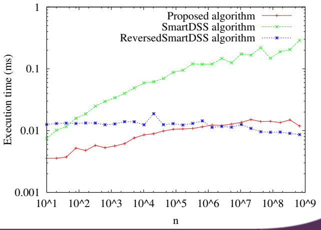
Implementation in C++ using DGtal

- comparison with [Said, Lachaud 2011] algorithms
- comparison and validation with classical arithmetical DSS recognition algorithm
- re-use experimental protocol of [Lachaud, Said 2012]
  - fixed maximal value N of the DSL period
  - fixed maximal length n of the DSS
  - randomly choose the DSL characteristics and the DSS length and position

$N = 10^6$



$N = 10^9$



Conclusion

Results

- a fast, integer only and easy to implement algorithm
- can be straightforwardly extended for digital straight lines with irrational characteristics

What next ?

- study the links with SmartDSS and ReversedSmartDSS
- use the Farey fan to solve other DSS problems (fast union, fast intersection, etc)