# A study of monodromy in the computation of multidimensional persistence

Andrea Cerri[1,4], **Marc Ethier**[2] and Patrizio Frosini[3,4]

[1] IMATI-CNR, Genova, Italia
[2] Département de mathématiques, Université de Sherbrooke Sherbrooke (Québec) Canada
[3] Dipartimento di Matematica, Università di Bologna, Italia
[4] ARCES, Università di Bologna, Italia

DGCI 2013
March 22, 2013

# Table of Contents

# Persistent homology

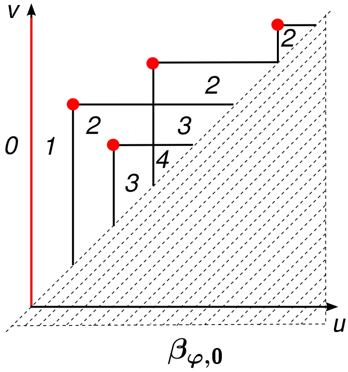
Let $X$ be a topological space and $\varphi : X \to \mathbb{R}$ a continuous function. The subsets $X_u = \{x \in X \mid \varphi(x) \leq u\}$ form a *filtration*. The inclusion $j^{(u,v)} : X_u \hookrightarrow X_v$ where $(u, v) \in \Delta^+ = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u < v\}$ induces for each $q$ an homomorphism $H_q(j^{(u,v)})$ between the homology modules $H_q(X_u)$ and $H_q(X_v)$.

# Persistent homology

Let $X$ be a topological space and $\varphi : X \to \mathbb{R}$ a continuous function. The subsets $X_u = \{x \in X \mid \varphi(x) \leq u\}$ form a *filtration*. The inclusion $j^{(u,v)} : X_u \hookrightarrow X_v$ where $(u, v) \in \Delta^+ = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u < v\}$ induces for each $q$ an homomorphism $H_q(j^{(u,v)})$ between the homology modules $H_q(X_u)$ and $H_q(X_v)$.
We define homology on a field $\mathbb{F}$ in order for the *persistent Betti numbers* $\beta_{\varphi,q}(u, v) = \dim \mathrm{im}\, H_q(j^{(u,v)})$ to entirely encapsulate persistence. $\beta_\varphi$ is further represented by its *persistence diagram* $\mathrm{Dgm}(\varphi)$.

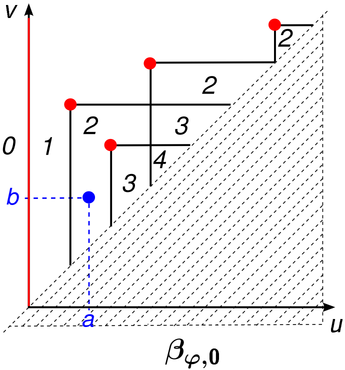
# Persistence diagrams

The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.

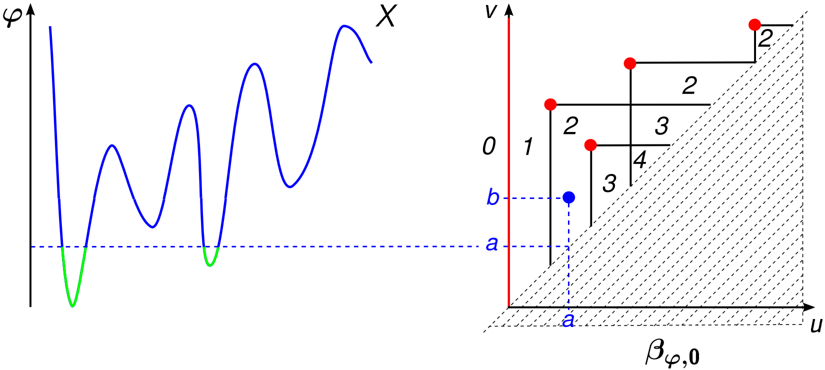
# Persistence diagrams

The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.

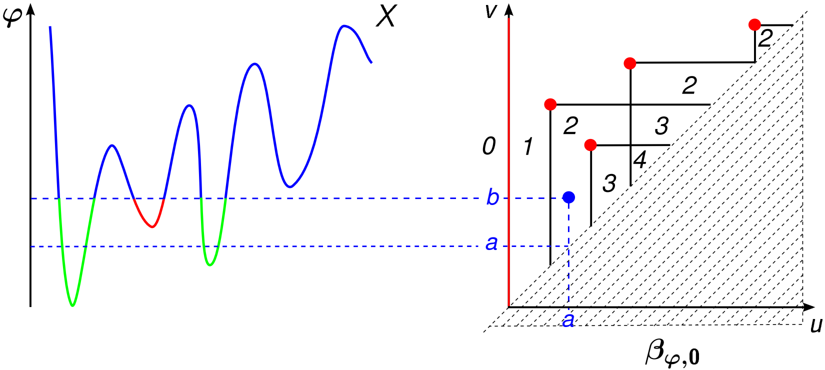
## Persistence diagrams

The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.



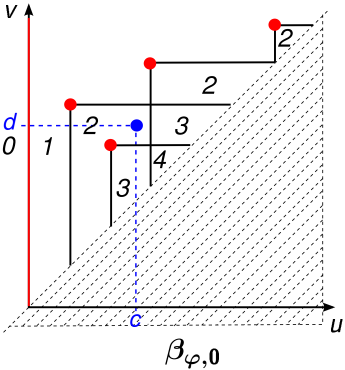
# Persistence diagrams

The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.



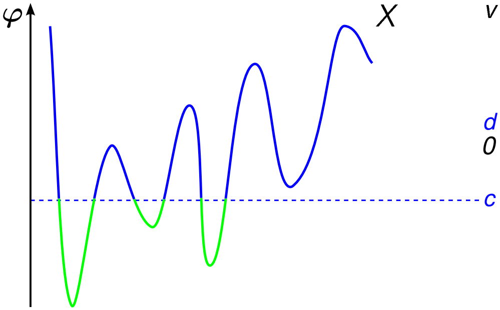
# Persistence diagrams

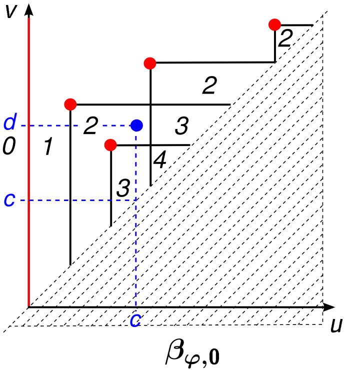
The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.

# Persistence diagrams

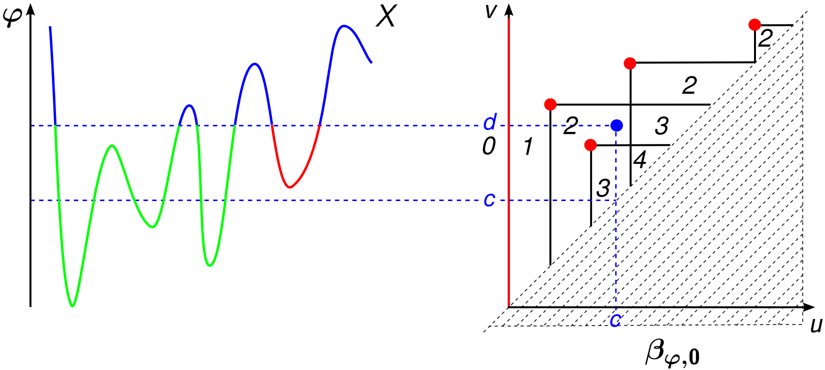
The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.



# Persistence diagrams

The persistence diagram $\mathrm{Dgm}(\varphi)$ is the *multiset* of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.



# Multidimensional persistence

For $\varphi : X \to \mathbb{R}^n$, we can consider the *multifiltration* formed by the subsets $X_u = \{x \in X \mid \varphi(x) \preceq u\}$, where $\preceq$ is the partial order on $\mathbb{R}^n$. We similarly obtain multidimensional persistent Betti numbers $\beta_{\varphi,q}(u, v)$.

# Multidimensional persistence

For $\varphi : X \to \mathbb{R}^n$, we can consider the *multifiltration* formed by the subsets $X_u = \{x \in X \mid \varphi(x) \preceq u\}$, where $\preceq$ is the partial order on $\mathbb{R}^n$. We similarly obtain multidimensional persistent Betti numbers $\beta_{\varphi,q}(u, v)$.
Why use multidimensional persistence?

# Multidimensional persistence

For $\varphi : X \to \mathbb{R}^n$, we can consider the *multifiltration* formed by the subsets $X_u = \{x \in X \mid \varphi(x) \preceq u\}$, where $\preceq$ is the partial order on $\mathbb{R}^n$. We similarly obtain multidimensional persistent Betti numbers $\beta_{\varphi,q}(u, v)$.
Why use multidimensional persistence?
It can be used to distinguish and compare noisy images, objects sampled by point clouds, and fuzzy sets.

# Image retrieval tolerant to domain perturbation

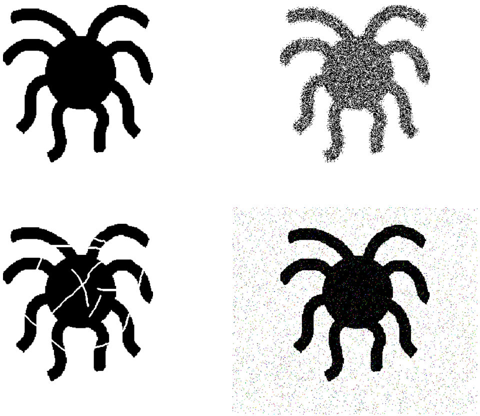
Let $X$ be a topological space, $K, K' \subset X$, and $\varphi : K \to \mathbb{R}^n$, $\varphi' : K' \to \mathbb{R}^n$ continuous filtering functions. If they represent point clouds or images subjected to noise, $K$ and $K'$ may differ in topology, making their comparison by means of persistent homology more problematic.

# Image retrieval tolerant to domain perturbation

Let $X$ be a topological space, $K, K' \subset X$, and $\varphi : K \to \mathbb{R}^n$, $\varphi' : K' \to \mathbb{R}^n$ continuous filtering functions. If they represent point clouds or images subjected to noise, $K$ and $K'$ may differ in topology, making their comparison by means of persistent homology more problematic.

However, extending $\varphi, \varphi'$ so that they take all $X$ as their domain, and substituting the sets $K, K'$ with appropriate functions $f_K, f_{K'} : X \to \mathbb{R}$ so that perturbations of the sets become perturbations of these functions, we can then use persistence to compare the functions $\Phi = (f_K, \varphi) : X \to \mathbb{R}^{n+1}$ and $\Phi' = (f_{K'}, \varphi') : X \to \mathbb{R}^{n+1}$.

# Examples of perturbed domains



Four binary images of an octopus. Last three correspond to the first one subjected to different kinds of noise.

# Choice of set distance function

The choice of $f_K$ depends on what deformation is expected. For small perturbations, sets are comparable using the Hausdorff distance, and we take as $f_K$ the distance from $K$ (in any norm). In presence of outlying points, sets can be compared using the symmetric difference pseudometric, in which case $f_K$ is taken as $\chi_K$ convolved with a ball.

# Choice of set distance function

The choice of $f_K$ depends on what deformation is expected. For small perturbations, sets are comparable using the Hausdorff distance, and we take as $f_K$ the distance from $K$ (in any norm). In presence of outlying points, sets can be compared using the symmetric difference pseudometric, in which case $f_K$ is taken as $\chi_K$ convolved with a ball.

See *Persistent Betti numbers for a noise tolerant shape-based approach to image retrieval*, P. Frosini and C. Landi (2012) for details.

# Choice of set distance function

The choice of $f_K$ depends on what deformation is expected. For small perturbations, sets are comparable using the Hausdorff distance, and we take as $f_K$ the distance from $K$ (in any norm). In presence of outlying points, sets can be compared using the symmetric difference pseudometric, in which case $f_K$ is taken as $\chi_K$ convolved with a ball.

See *Persistent Betti numbers for a noise tolerant shape-based approach to image retrieval*, P. Frosini and C. Landi (2012) for details.

This method requires the use of multidimensional persistence.

# Computation of multidimensional persistence

Computing the multidimensional persistence diagram for a function $\varphi : X \to \mathbb{R}^n$ reduces to the computation of one-dimensional persistence diagrams for a parametrized family of functions $\varphi_{(\vec{m},\vec{b})} : X \to \mathbb{R}$ defined as

$$\varphi_{(\vec{m},\vec{b})}(x) = \min_i m_i \cdot \max_i \left\{ \frac{\varphi_i(x) - b_i}{m_i} \right\},$$

where $(\vec{m}, \vec{b})$ varies in the space

$$\mathrm{Adm}_n = \left\{ (\vec{m}, \vec{b}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall i \; m_i > 0, \sum_i m_i = 1, \sum_i b_i = 0 \right\}.$$

# Computation of multidimensional persistence

Computing the multidimensional persistence diagram for a function $\varphi : X \to \mathbb{R}^n$ reduces to the computation of one-dimensional persistence diagrams for a parametrized family of functions $\varphi_{(\vec{m},\vec{b})} : X \to \mathbb{R}$ defined as

$$\varphi_{(\vec{m},\vec{b})}(x) = \min_i m_i \cdot \max_i \left\{ \frac{\varphi_i(x) - b_i}{m_i} \right\},$$

where $(\vec{m}, \vec{b})$ varies in the space

$$\mathrm{Adm}_n = \left\{ (\vec{m}, \vec{b}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall i \ m_i > 0, \sum_i m_i = 1, \sum_i b_i = 0 \right\}.$$

$(\vec{m}, \vec{b}) \in \mathrm{Adm}_n$ corresponds to a line $r_{(\vec{m},\vec{b})}$ of $\mathbb{R}^n$ whose distinct points $u = \sigma\vec{m} + \vec{b}$, $v = \tau\vec{m} + \vec{b}$, $\sigma, \tau \in \mathbb{R}$ are comparable by $\prec$.

# Tracking cornerpoints

## Definition

*The pair* $(\vec{m}, \vec{b}) \in Adm_n$ *is said to be* singular *if at least one proper cornerpoint of* $Dgm(\varphi_{(\vec{m},\vec{b})})$ *has multiplicity strictly greater than 1. Otherwise it is called* regular*. The set of regular pairs shall be denoted* $Adm_n^*$*. Moreover,* $\varphi$ *is said to be* normal *if its set of singular pairs is discrete.*

## Theorem

*Let* $\varphi : X \to \mathbb{R}^n$ *be a normal filtering function and* $I$ *be the closed interval* $[0, 1]$*. For every continuous path* $\gamma : I \to Adm_n^*(\varphi)$ *and every proper cornerpoint* $p \in Dgm(\varphi_{\gamma(0)})$*, there exists a continuous function* $c : I \to \Delta^+ \cup \{\Delta\}$ *such that* $c(0) = p$ *and* $c(t) \in Dgm(\varphi_{\gamma(t)})$ *for all* $t \in I$*. Furthermore, if there is no* $t \in I$ *such that* $c(t) = \Delta$*,* $c$ *is the only such continuous function.*

# Correspondence of cornerpoints to admissible pair

We might think that there exists a correspondence between the pair $(\vec{m}, \vec{b}) \in \mathrm{Adm}_n^*(\varphi)$ and each cornerpoint of the diagram $\mathrm{Dgm}(\varphi_{(\vec{m},\vec{b})})$.

# Correspondence of cornerpoints to admissible pair

We might think that there exists a correspondence between the pair $(\vec{m}, \vec{b}) \in \mathrm{Adm}_n^*(\varphi)$ and each cornerpoint of the diagram $\mathrm{Dgm}(\varphi_{(\vec{m},\vec{b})})$.
However, this is not the case: the correspondence depends on the path followed.

# Example

## Example (Nontrivial monodromy)

Consider the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined on the plane in the following way: $\varphi_1(x, y) = x$, and

$$\varphi_2(x, y) = \begin{cases} -x & \text{if } y = 0 \\ -x + 1 & \text{if } y = 1 \\ -2x & \text{if } y = 2 \\ -2x + \frac{5}{4} & \text{if } y = 3 \end{cases},$$

$\varphi_2(x, y)$ then being extended linearly for every $x$ on the segment joining $(x, 0)$ with $(x, 1)$, $(x, 1)$ with $(x, 2)$, and $(x, 2)$ to $(x, 3)$. On the half-lines $\{(x, y) \in \mathbb{R}^2 \mid y < 0\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y > 3\}$, $\varphi_2$ is then being taken with constant slope $-1$ in the variable $y$.

# Example

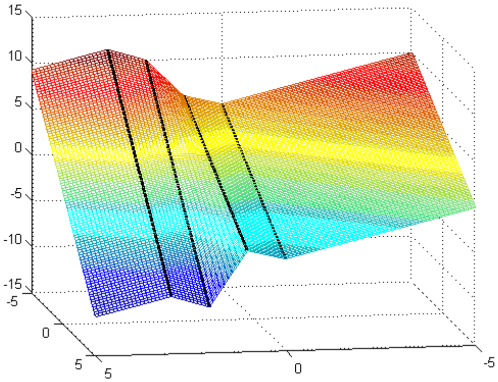



Figure : Function $\varphi_2$ of previous example. Depth is $x$, width is $y$.

# Monodromy

Let $p : \widetilde{X} \to X$ a covering map onto the topological space $X$, and let $x \in X$. In algebraic topology, we refer to as *monodromy* the phenomenon by which, for a loop $\gamma : I \to X$ where $\gamma(0) = \gamma(1) = x$, and for $\tilde{x} \in p^{-1}(x)$ an element of the fibre of $x$, the associated continuous path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \tilde{x}$ and $p \circ \tilde{\gamma} = \gamma$ might not be such that $\tilde{\gamma}(1) = \tilde{x}$.

# Monodromy

Let $p : \widetilde{X} \to X$ a covering map onto the topological space $X$, and let $x \in X$. In algebraic topology, we refer to as *monodromy* the phenomenon by which, for a loop $\gamma : I \to X$ where $\gamma(0) = \gamma(1) = x$, and for $\tilde{x} \in p^{-1}(x)$ an element of the fibre of $x$, the associated continuous path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \tilde{x}$ and $p \circ \tilde{\gamma} = \gamma$ might not be such that $\tilde{\gamma}(1) = \tilde{x}$.

In other words, as we turn around a singularity, it may be necessary to define applications on the cover $\widetilde{X}$ of $X$ in order to guarantee their continuity.

# Consequences

We are currently studying distances between two-dimensional persistence diagrams that follow families of matchings between their cornerpoints as the latter move continuously in the space $\Delta^+ \cup \{\Delta\}$ under changes of admissible pair $(\vec{m}, \vec{b})$.

# Consequences

We are currently studying distances between two-dimensional persistence diagrams that follow families of matchings between their cornerpoints as the latter move continuously in the space $\Delta^+ \cup \{\Delta\}$ under changes of admissible pair $(\vec{m}, \vec{b})$.
However, because of the monodromy property, the cost of a matching depends not only on the pair $(\vec{m}, \vec{b})$ at which we find ourselves, but also on the path taken to reach it.

# Consequences

We are currently studying distances between two-dimensional persistence diagrams that follow families of matchings between their cornerpoints as the latter move continuously in the space $\Delta^+ \cup \{\Delta\}$ under changes of admissible pair $(\vec{m}, \vec{b})$.

However, because of the monodromy property, the cost of a matching depends not only on the pair $(\vec{m}, \vec{b})$ at which we find ourselves, but also on the path taken to reach it.

We therefore believe that continuous families of matchings should take as parameter not the admissible pair $(\vec{m}, \vec{b}) \in \mathrm{Adm}_2$, but rather an element of the fibre of $(\vec{m}, \vec{b})$ in a suitable covering space.