A study of monodromy in the computation of multidimensional persistence

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$$(u, v) \in \Delta^+ = \{ (u, v) \in \mathbb{R} \times \mathbb{R} \mid u < v \}$$

induces for each $q$ an homomorphism $H_q(j^{(u,v)})$ between the homology modules $H_q(X_u)$ and $H_q(X_v)$. 
Persistent homology

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We define homology on a field $\mathbb{F}$ in order for the persistent Betti numbers $\beta_\varphi,q(u, v) = \dim \text{im} H_q(j^{(u,v)})$ to entirely encapsulate persistence. $\beta_\varphi$ is further represented by its persistence diagram $\text{Dgm}(\varphi)$. 
Persistence diagrams

The persistence diagram $Dgm(\varphi)$ is the multiset of its proper cornerpoints and cornerpoints at infinity repeated a number of times corresponding to their multiplicity, with the diagonal $\Delta = \{(u, v) \in \mathbb{R} \times \mathbb{R} \mid u = v\}$ taken with infinite multiplicity.
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Multidimensional persistence

For $\varphi : X \to \mathbb{R}^n$, we can consider the *multifiltration* formed by the subsets $X_u = \{ x \in X \mid \varphi(x) \preceq u \}$, where $\preceq$ is the partial order on $\mathbb{R}^n$. We similarly obtain multidimensional persistent Betti numbers $\beta_{\varphi,q}(u, v)$. 

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Why use multidimensional persistence?
It can be used to distinguish and compare noisy images, objects sampled by point clouds, and fuzzy sets.
Image retrieval tolerant to domain perturbation

Let $X$ be a topological space, $K, K' \subset X$, and $\varphi : K \to \mathbb{R}^n$, $\varphi' : K' \to \mathbb{R}^n$ continuous filtering functions. If they represent point clouds or images subjected to noise, $K$ and $K'$ may differ in topology, making their comparison by means of persistent homology more problematic.
Let $X$ be a topological space, $K, K' \subset X$, and $\varphi : K \to \mathbb{R}^n$, $\varphi' : K' \to \mathbb{R}^n$ continuous filtering functions. If they represent point clouds or images subjected to noise, $K$ and $K'$ may differ in topology, making their comparison by means of persistent homology more problematic. However, extending $\varphi, \varphi'$ so that they take all $X$ as their domain, and substituting the sets $K, K'$ with appropriate functions $f_K, f_{K'} : X \to \mathbb{R}$ so that perturbations of the sets become perturbations of these functions, we can then use persistence to compare the functions $\Phi = (f_K, \varphi) : X \to \mathbb{R}^{n+1}$ and $\Phi' = (f_{K'}, \varphi') : X \to \mathbb{R}^{n+1}$. 
Examples of perturbed domains

Four binary images of an octopus. Last three correspond to the first one subjected to different kinds of noise.
Choice of set distance function

The choice of $f_K$ depends on what deformation is expected. For small perturbations, sets are comparable using the Hausdorff distance, and we take as $f_K$ the distance from $K$ (in any norm). In presence of outlying points, sets can be compared using the symmetric difference pseudometric, in which case $f_K$ is taken as $\chi_K$ convolved with a ball.
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See *Persistent Betti numbers for a noise tolerant shape-based approach to image retrieval*, P. Frosini and C. Landi (2012) for details.
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Computation of multidimensional persistence

Computing the multidimensional persistence diagram for a function \( \varphi : X \to \mathbb{R}^n \) reduces to the computation of one-dimensional persistence diagrams for a parametrized family of functions \( \varphi(\vec{m}, \vec{b}) : X \to \mathbb{R} \) defined as

\[
\varphi(\vec{m}, \vec{b})(x) = \min_i m_i \cdot \max_i \left\{ \frac{\varphi_i(x) - b_i}{m_i} \right\},
\]

where \((\vec{m}, \vec{b})\) varies in the space

\[
\text{Adm}_n = \left\{ (\vec{m}, \vec{b}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall i \; m_i > 0, \sum_i m_i = 1, \sum_i b_i = 0 \right\}.
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$(\vec{m}, \vec{b}) \in \text{Adm}_n$ corresponds to a line $r_{(\vec{m}, \vec{b})}$ of $\mathbb{R}^n$ whose distinct points $u = \sigma \vec{m} + \vec{b}, \nu = \tau \vec{m} + \vec{b}, \sigma, \tau \in \mathbb{R}$ are comparable by $\prec$. 
Tracking cornerpoints

**Definition**

The pair \((\vec{m}, \vec{b})\) \(\in Adm_n\) is said to be singular if at least one proper cornerpoint of \(Dgm(\varphi_{(\vec{m}, \vec{b})})\) has multiplicity strictly greater than 1. Otherwise it is called regular. The set of regular pairs shall be denoted \(Adm^*_n\). Moreover, \(\varphi\) is said to be normal if its set of singular pairs is discrete.

**Theorem**

Let \(\varphi : X \rightarrow \mathbb{R}^n\) be a normal filtering function and \(I\) be the closed interval \([0, 1]\). For every continuous path \(\gamma : I \rightarrow Adm^*_n(\varphi)\) and every proper cornerpoint \(p \in Dgm(\varphi_{\gamma(0)})\), there exists a continuous function \(c : I \rightarrow \Delta^+ \cup \{\Delta\}\) such that \(c(0) = p\) and \(c(t) \in Dgm(\varphi_{\gamma(t)})\) for all \(t \in I\). Furthermore, if there is no \(t \in I\) such that \(c(t) = \Delta\), \(c\) is the only such continuous function.
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$\text{Dgm}(\varphi_{(\vec{m}, \vec{b})})$. 
However, this is not the case: the correspondence depends on the path followed.
Example (Nontrivial monodromy)

Consider the function \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined on the plane in the following way: \( \varphi_1(x, y) = x \), and

\[
\varphi_2(x, y) = \begin{cases} 
-x & \text{if } y = 0 \\
-x + 1 & \text{if } y = 1 \\
-2x & \text{if } y = 2 \\
-2x + \frac{5}{4} & \text{if } y = 3
\end{cases}
\]

\( \varphi_2(x, y) \) then being extended linearly for every \( x \) on the segment joining \((x, 0)\) with \((x, 1)\), \((x, 1)\) with \((x, 2)\), and \((x, 2)\) to \((x, 3)\). On the half-lines \( \{(x, y) \in \mathbb{R}^2 \mid y < 0\} \) and \( \{(x, y) \in \mathbb{R}^2 \mid y > 3\} \), \( \varphi_2 \) is then being taken with constant slope \(-1\) in the variable \( y \).
Example

**Figure**: Function $\varphi_2$ of previous example. Depth is $x$, width is $y$. 
Let $p : \tilde{X} \to X$ a covering map onto the topological space $X$, and let $x \in X$. In algebraic topology, we refer to as monodromy the phenomenon by which, for a loop $\gamma : I \to X$ where $\gamma(0) = \gamma(1) = x$, and for $\tilde{x} \in p^{-1}(x)$ an element of the fibre of $x$, the associated continuous path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \tilde{x}$ and $p \circ \tilde{\gamma} = \gamma$ might not be such that $\tilde{\gamma}(1) = \tilde{x}$. 
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In other words, as we turn around a singularity, it may be necessary to define applications on the cover $\tilde{X}$ of $X$ in order to guarantee their continuity.
Consequences

We are currently studying distances between two-dimensional persistence diagrams that follow families of matchings between their cornerpoints as the latter move continuously in the space \( \Delta^+ \cup \{\Delta\} \) under changes of admissible pair \((\vec{m}, \vec{b})\).
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