On the degree sequences of uniform hypergraphs

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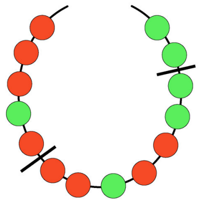


Definitions

Projections of a binary matrix

Given a binary matrix we can define its horizontal and vertical **projections** as shown below:

	$V =$	4	3	3	0	2	3	3	2
	4	1	0	0	0	1	1	1	0
	3	1	0	0	0	0	1	1	0
$H =$	5	1	1	1	0	0	1	0	1
	2	0	0	1	0	0	0	1	0
	5	1	1	1	0	1	0	0	1
	1	0	1	0	0	0	0	0	0

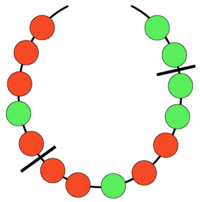
Consistency and Reconstruction problems

Consistency (H,V)
Input: two integer vectors H and V
Question: does there exist a binary matrix A consistent with H and V?

Reconstruction (H,V)
Input: two integer vectors H and V
Task: reconstruct a binary matrix A consistent with H and V, if possible, otherwise give failure.

We challenge these two problems after imposing the further constraints that A has to have different rows

These problems are related to the *characterization of the degree sequences of uniform hypergraphs*



The consistency problem

Ryser's conditions

Ryser gave the following conditions to solve the consistency problem for a generic matrix *m x n*:

1. for each $1 \leq i \leq m$ and $1 \leq j \leq n$, it holds
$h_i \leq n$ and $v_j \leq m$;
2. for all $1 \leq i \leq m$ and $1 \leq j \leq n$, it holds
$\Sigma h_i = \Sigma v_j$;
3. *for all* $1 \leq i \leq m$, it holds,
$\Sigma_{j=1..i} h_j \geq \Sigma_{j=1..i} v_j'$;
V' being the conjugate sequence of V

These three conditions can be tested in polynomial time.

We investigate if these conditions are also suitable for matrices that contain no equal rows. We restrict our study to the case when the two vectors of projections are **homogeneous.**

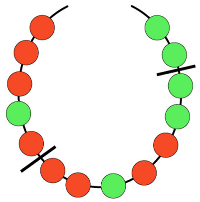
Homogeneous projections

An useful result:
if the horizontal and vertical projections are homogeneous, then **conditions *1*** and ***2*** are enough to solve the consistency problem.

A third simple condition

3 bis. Consistency has a negative answer if
$m \geq binomial\ (n,h)$

Theorem:
Conditions 1*, *2*,** and ***3 bis are necessary and sufficient in order to solve *Consistency (H, V)*, with *H* and *V* homogeneous, and the solution having no equal rows



Reconstructing binary matrices with different rows

Arranging the rows of a matrix into classes

Each row of a binary matrix can be considered as a word in a binary alphabet.
Let us recall the following definitions:

(Binary) necklace: an equivalence class $[u]$ of binary words u under cyclic shift.

Lyndon word: an aperiodic necklace.

Matrix of a necklace: the matrix $M(u)$ obtained by superposing the element of the necklace $[u]$

Properties of $M(u)$, with $u=v^k$:
- $M(u)$ has n/k rows
- $M(u)$ has homogeneous projections with $v = h/k$
- $M(1^h0^{n-h})$ has n rows, and they can be arranged into submatrices with homogeneous projections such that the vertical projections are minimal and equal to $h/gcd\{n,h\}$

J. Sawada defined a CAT algorithm to generate necklaces and Lyndon words

Reconstructing binary matrices with different rows

Sketch of the reconstruction algorithm

Input: two homogeneous vectors H and V
Output: a matrix A with no equal rows and compatible with H and V

Step 1: Let compute the sequence $d_0 = 1 < d_1 < d_2 < \ldots < d_t$ of the common divisors of n and h. *Let* $i=0$;

Step 2: By applying *GenLyndon(n', h),* generate a Lyndon word u of length $n'=n/d_i$ and append $M(u)$ it to A, if it does not overcame the dimension m, otherwise set $i=i+1$ and goto *Step 2*. If the rows of A are m then output A.

0	**0**	**0**	**0**	**1**	**1**
1	0	0	0	0	1
1	1	0	0	0	0
0	1	1	0	0	0
0	0	1	1	0	0
0	0	0	1	1	0
0	**0**	**0**	**1**	**0**	**1**
1	0	0	0	1	0
0	1	0	0	0	1
1	0	1	0	0	0
0	1	0	1	0	0
0	0	1	0	1	0

0	**0**	**0**	**0**	**1**	**1**
1	0	0	0	0	1
1	1	0	0	0	0
0	1	1	0	0	0
0	0	1	1	0	0
0	0	0	1	1	0
0	**0**	**0**	**1**	**0**	**1**
1	0	0	0	1	0
0	1	0	0	0	1
1	0	1	0	0	0
0	1	0	1	0	0
0	0	1	0	1	0
0	**0**	**1**	**0**	**0**	**1**
1	0	0	1	0	0
0	1	0	0	1	0

An example for a *15 x 6* matrix $\boldsymbol{h = 2}$ and $\boldsymbol{v = 5}$

Some care is needed if no suitable Lyndon words are found: the Lyndon word 1^h0^{n-h} can help!

Open problems

- generalize the algorithm to instances having different vertical projections

- generalize the algorithm taking care of the ***inclusion*** of rows with homogeneous vertical projections and different rows projections

- final generalization in the case of inclusion with generic rows and columns projections