Generalized Simple Surface Points

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Joint work with:

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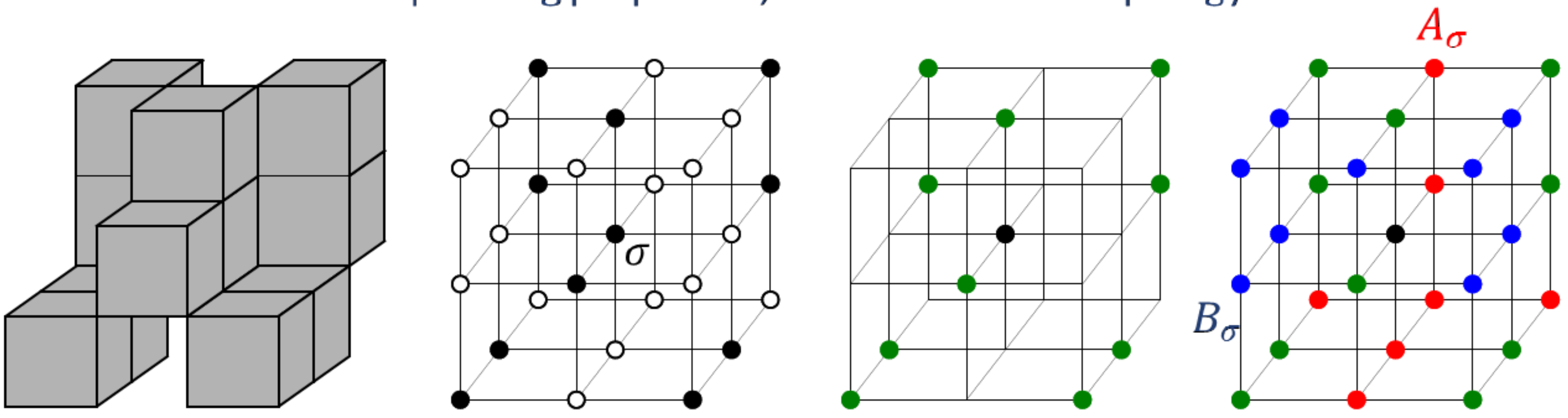
Introduction

This paper deals with the notion of discrete surface considered as

- "thin" subsets of voxels (in the usual grid $\mathbb{Z}^3$)
- with suitable separating properties, as in continuous topology



The first definition of a discrete surface (Morgenthaler & Rosenfeld 1981):

A simple (closed) surface is a set $S \subseteq \mathbb{Z}^3$ consisting entirely of simple surface points.

A voxel $\sigma \in S$ is a simple surface point if

a) just a component of $N_{26}(\sigma) \cap S - \{\sigma\}$ is *adjacent* to σ.

b) S is thin at σ: $N_{26}(\sigma) - S$ contains just two components, A_σ and B_σ, *adjacent* to σ.

c) each voxel $\tau \in S$ *adjacent* to σ is *adjacent* to both A_σ and B_σ.

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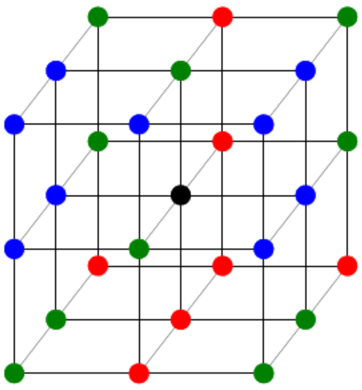

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Theorem. Any connected simple surface $S \subseteq \mathbb{Z}^3$ is a strong separating set; that is, separates its background $\mathbb{Z}^3 - S$ into two components, A and B, and each voxel $\sigma \in S$ is adjacent to both of these components.

Introduction

Morgenthaler & Rosefenld definition is simple, but too restrictive

- it is just given in terms of the usual adjacency pairs between voxels and subsets
- the notion of surface point is local (26-neighbourhood)
- for the pair (26,6), up to rotations and symmetries, only 13 configurations may appear in the 26-neighbourhood of a voxel

Other definitions in literature:	# configurations for the (26,6)-adjacency
▪ Strong surfaces (Malgouyres & Bertrand, 1999)	705
▪ Simplicity surfaces (Couprie & Bertrand, 1998)	736
▪ $(k,\overline{k})$-surfaces (Ciria et al., 2009,2012)	10.580
▪ … and many other	

Drawback: all of them require some additional global or local property

Our goal: to generalize the notion of simple surface point, still using only the adjacency relations between voxels, and characterize these surfaces as subsets of simple surface points.

Generalized simple surface points

A voxel $\sigma \in S$ is a simple surface point if

a) just a component of $N_{26}(\sigma) \cap S - \{\sigma\}$ is *adjacent* to σ. [Redundant, Kong 1985]

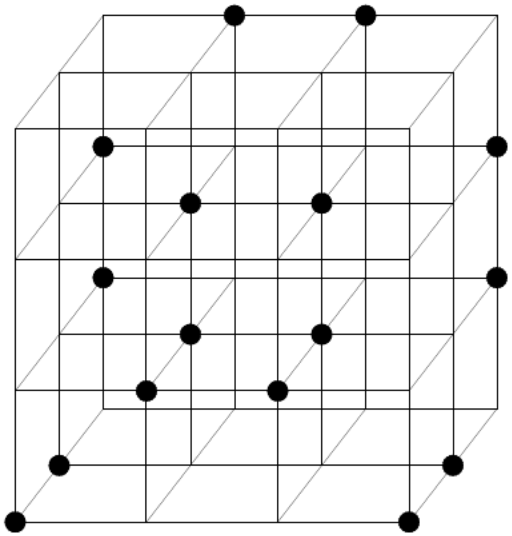
b) S is thin at σ. [Keeped to ensure suitable separation properties]

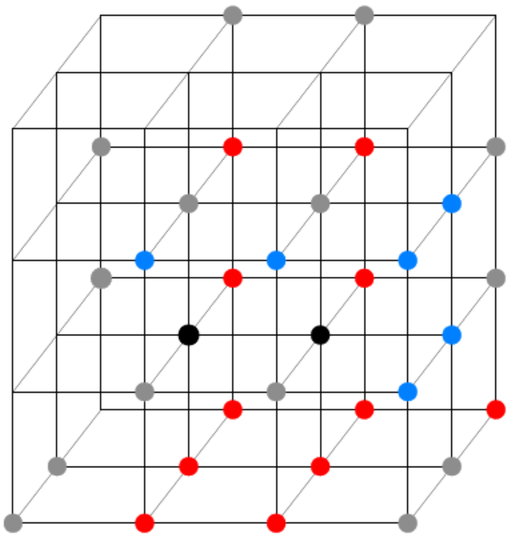
c) each voxel $\tau \in S$ adjacent to σ is *adjacent* to both A_σ and B_σ. [to be weakened]

Example for

$(k, \overline{k}) = (26,6)$

a simplicity 26-surface





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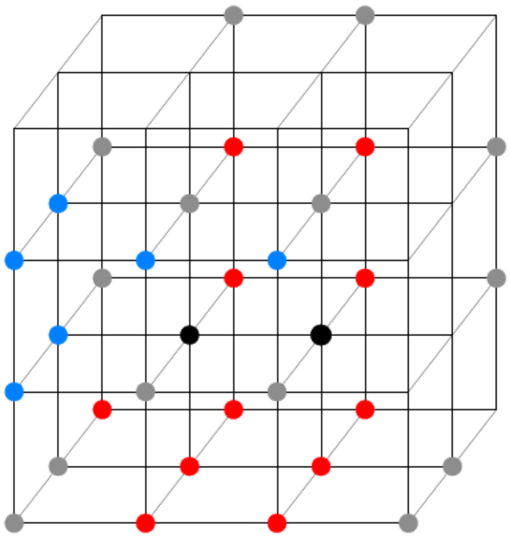
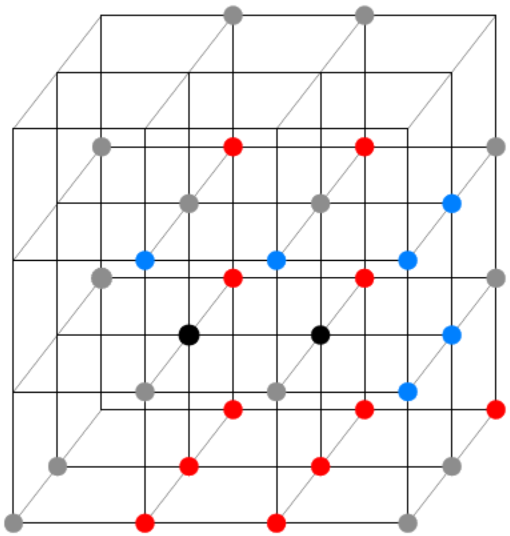
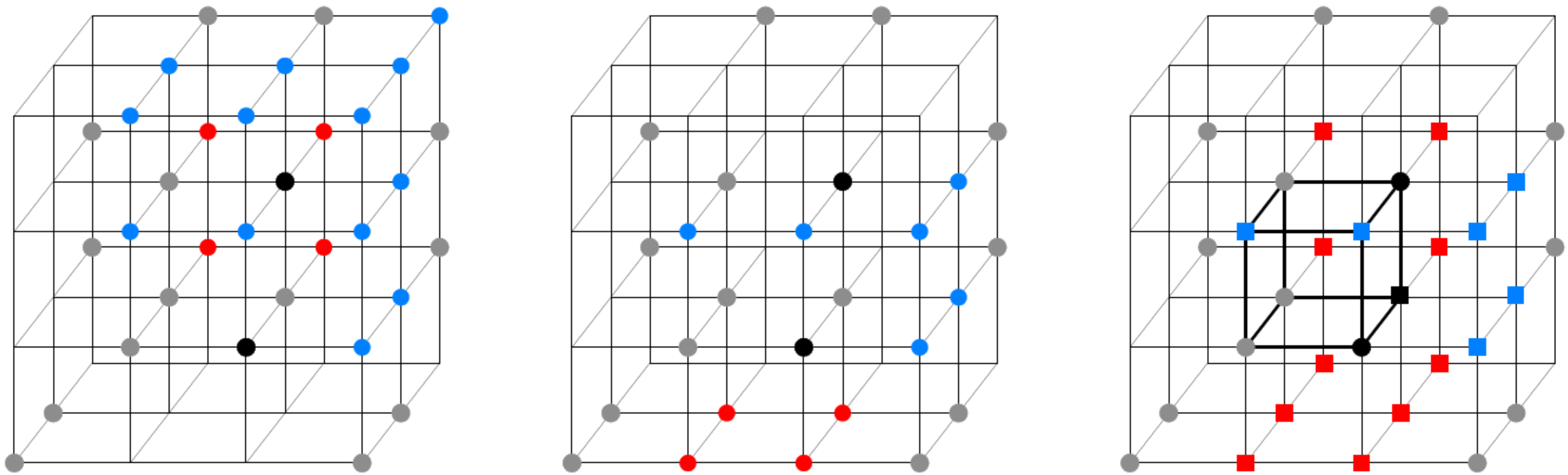




Def. Let $O \subseteq \mathbb{Z}^3$ be $\overline{k}$-thin at two 26-adjacent voxels $\sigma, \tau \in O$. The voxel σ is $\overline{k}_6$-linked to τ if the $\overline{k}$-components A_σ and B_σ meet $A_\tau \cup B_\tau$.

Generalized simple surface points

- The $\overline{k}_6$-linking property is too restrictive for voxels which are strictly 18- or 26-adjacent

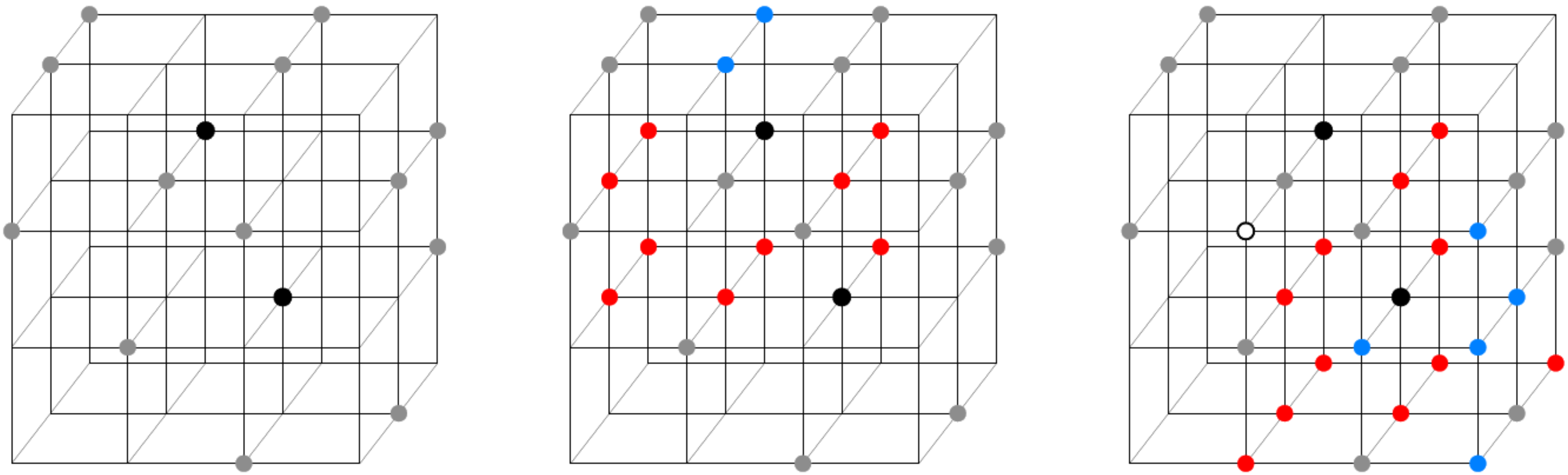


Def. A voxel $\sigma \in O$ is $\overline{k}_{18}$-linked to τ if there is a unit cube containing a $\overline{k}_6$-path from σ to τ. A $\overline{k}_6$-path is a sequence of voxels $\sigma_0, \sigma_1, \ldots, \sigma_n$ such that σ_{i-1} is $\overline{k}_6$-linked to σ_i, for $1 \leq i \leq n$.

Generalized simple surface points

- The $\overline{k}_{18}$-linking property is too restrictive for voxels which are strictly 26-adjacent

Example: a (26,6)-surface in the sense of [Ciria et al., 2009] but not a simplicity surface



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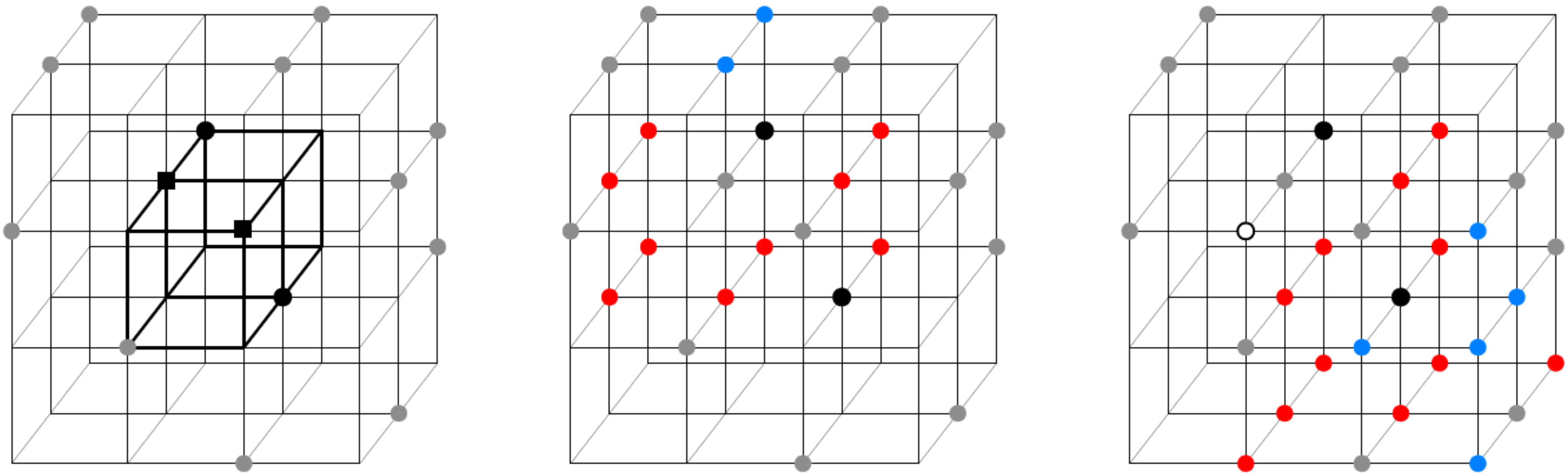


Def. A voxel $\sigma \in O$ is $\overline{k}_{26}$-linked to τ if a $\overline{k}_{6}$-path from σ to τ is found in the union of two unit cubes.

Surface points and simple surfaces

Given an adjacency pair $(k, \overline{k})$ on $\mathbb{Z}^3$, where $k, \overline{k} \in \{6, 18, 26\}$ …

Def. A voxel $\sigma \in O$ is a simple $(k, \overline{k})$-surface point if

1. O is $\overline{k}$-thin at σ, and also at each voxel $\tau \in O$ which is k-adjacent to it

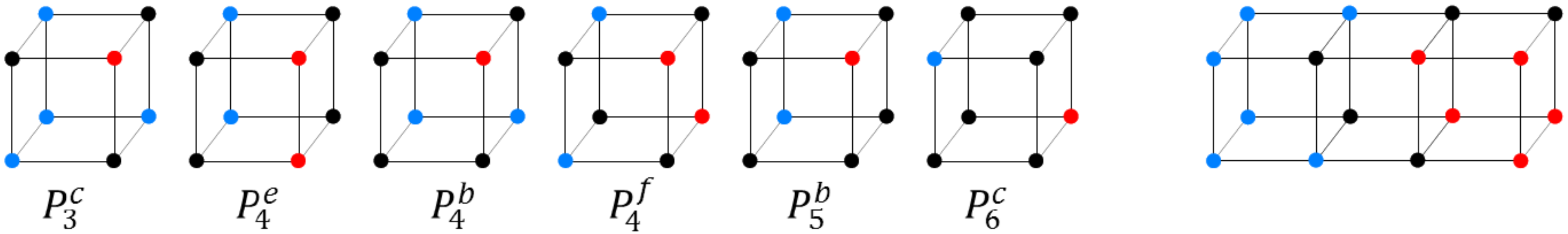
2a. σ is $\overline{k}_6$-linked to each voxel $\tau \in O$ which is 6-adjacent to it.

2b. If $k \geq 18$, σ is $\overline{k}_{18}$-linked to each voxel $\tau \in O$ which is (strictly) 18-adjacent to it.

2c. If $k = 26$, σ is $\overline{k}_{26}$-linked to each voxel $\tau \in O$ which is (strictly) 26-adjacent to it.

[2. σ is $\overline{k}_t$-linked to each voxel $\tau \in O$ which is t-adjacent to it, for $k \geq t \in \{6,18,26\}$]

Def. A simple $(k, \overline{k})$-surface is a subset $S \subseteq \mathbb{Z}^3$ consisting entirely of simple $(k, \overline{k})$-surface points and such that if $K - S$ is not $\overline{k}$-connected, for a unit cube K, then it meets A_σ and B_σ for each voxel $\sigma \in K \cap S$.

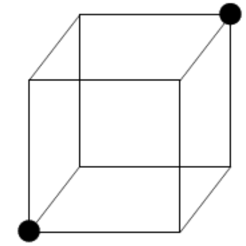


Forbidden patterns

Prop. 9 & 10. Let S be a simple $(k, \overline{k})$-surface. Then, depending on the value of k and $\overline{k}$, the following patterns cannot appear in S.

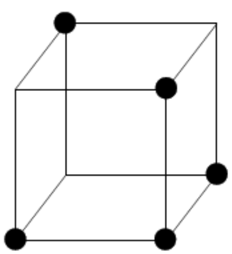
Proof. Case analysis.

$k = 26$

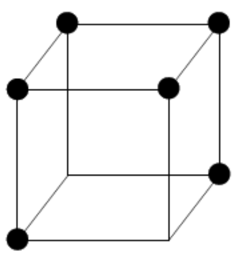


P_2^c

$\overline{k} = 6$

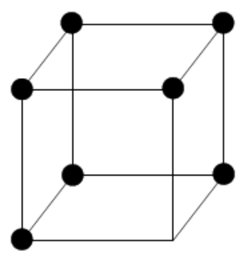


P_5^c



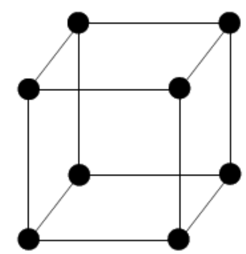
P_6^b

$\overline{k} = 18$



P_7

$\overline{k} = 26$



P_8

Main result

Next steps:

- Check that simple surfaces are truly discrete surfaces: they are strong separating sets.
- The family of simple surfaces contains some other families of surfaces in literature (strong and simplicity surfaces).

Both attained by:

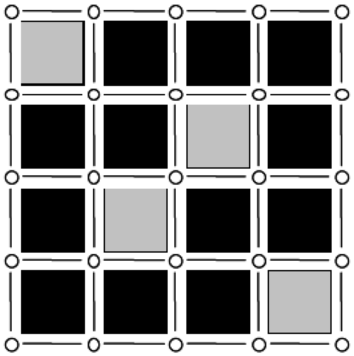
Main Theorem: For $(k,\overline{k}) \neq (6,6)$, a subset $S \subseteq \mathbb{Z}^3$ is a simple $(k,\overline{k})$-surface if and only if it is a strong $\overline{k}$-separating $(k,\overline{k})$-surface in the sense of [Ciria et al. 2009, 2012].

- [Ciria et al. 2009] Defines the family of $(k,\overline{k})$-surfaces by means of continuous analogues and, using previous works, show that it contains the families of strong and simplicity surfaces.
- [Ciria et al. 2012] Gives a local characterization of $(k,\overline{k})$-surfaces using plates (small pieces of surfaces within a unit cube) that
  - provides a geometrical interpretation
  - allows to count the number of configurations in the 26-neighbourhood of a voxel

Continuous Analogues

Discrete model: a polyhedral complex

R^3: the tiling of the Euclidean space $\mathbb{R}^3$ by unit cubes



Only n-cells represent pixels/voxels:

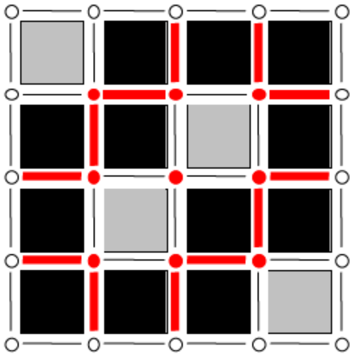
$$O \subseteq \mathrm{cell}_3(R^3) \cong \mathbb{Z}^3$$

Lighting function:

$$f: \wp(\mathrm{cell}_3(R^3)) \times R^3 \to \{0,1\}$$

associates each object O to

$$\{\alpha \in R^3 \mid f(O,\alpha) = 1\}$$



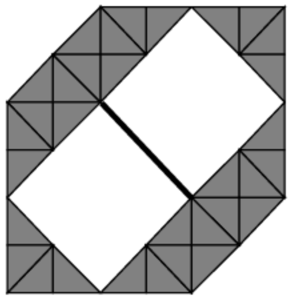
If $f(O,\alpha) = 1$ we say that α is lighted for O.

A digital space is a pair (R^3, f):

- a discrete model for images: R^3
- and a continuous interpretation, f, of each image

Continuous analogue:

$$\mathcal{A}_O \triangleleft R^{3(1)}$$



To ensure that continuous analogue are not contradictory with our topological intuition, lighting functions must satisfy FIVE axioms.

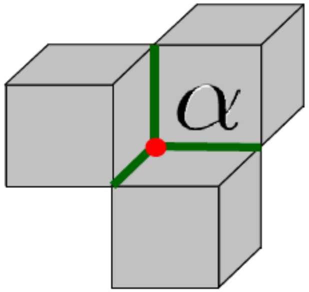
Digital surfaces

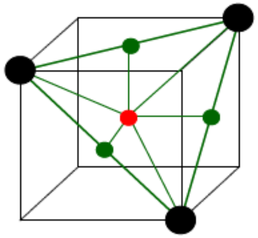
Since continuous analogues intend to be a continuous interpretation of digital objects…

…properties of digital objects may be defined as those of their corresponding continuous analogues

Def.: Let (R^3, f) be a digital space. A digital object $S \subseteq \mathrm{cell}_3(R^3)$ is a digital surface in (R^3, f) (or, an f-surface) if its continuous analogue $\mathcal{A}_S$ is a combinatorial surface (without boundary); that is, if the link $\mathrm{lk}(c(\alpha), \mathcal{A}_S)$ of each vertex $c(\alpha) \in \mathcal{A}_S$ is a 1-sphere.

Recall: $\mathrm{lk}(c(\alpha), \mathcal{A}_S) = \{X \in \mathcal{A}_S | \exists Y \in \mathcal{A}_S$ such that $c(\alpha) < X$, Y<X, and $c(\alpha) \nless Y\}$





$(k,\overline{k})$-spaces

We are only interested in surfaces of digital spaces for which the connectedness of the continuous analogues is coherent with the connectedness induced by an adjacency pair $(k,\overline{k})$ on the grid $\mathbb{Z}^3$, $k,\overline{k} \in \{6,18,26\}$.

Def.: (R^3, f) is said to be a $(k,\overline{k})$-space if for any object O:

- The continuous analogue $|\mathcal{A}_O|$ is a connected polyhedron iff O is k-connected
- The difference $|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$ is connected iff $\mathbb{Z}^3 - O$ is $\overline{k}$-connected

Th.: (Ciria et al., 2009) There is a single (6,6)-space, and it does not have any surface.

Th.: (Ciria et al., 2009) Let $S \subseteq \mathbb{Z}^3$ be a digital surface in a $(k,\overline{k})$-space, with $(k,\overline{k}) \neq (6,6)$, then S is also a surface in the universal $(k,\overline{k})$-space $(R^3, f_{k\overline{k}})$.

The universal $(k, \overline{k})$-space

Let K_δ be the set of voxels that have the cell $\delta \in R^3$ as a face

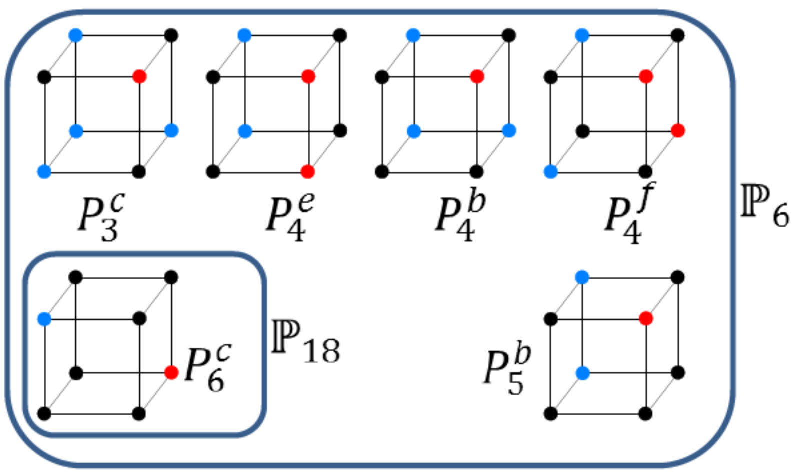
Def.: Given $O \subseteq \mathbb{Z}^3$ and $\delta \in O$, $f_{k\overline{k}}(O, \delta) = 1$ if

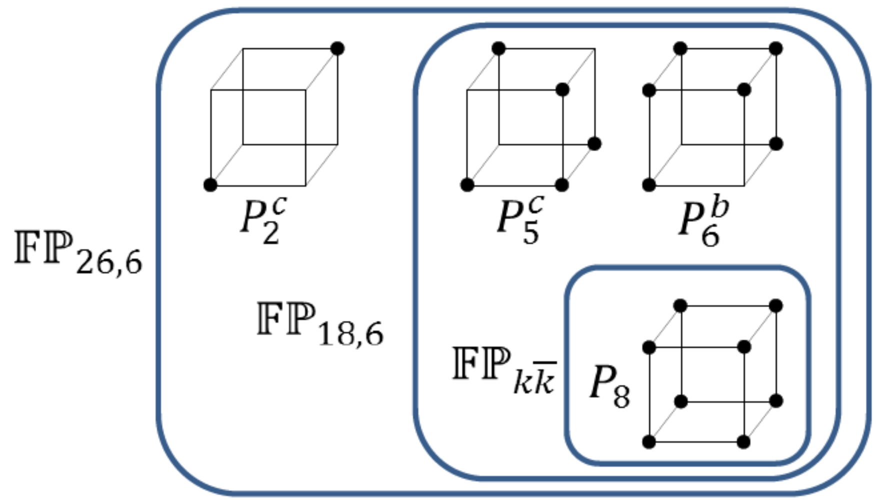
a. $K_\delta \subseteq O$ contains all voxels that have δ as a face.

b. δ is a vertex and $K_\delta \cap O$ corresponds, up to rotations and symmetries, to some pattern in the set $\mathbb{P}_{\overline{k}} \cup \mathbb{FP}_{k\overline{k}}$ (where $\mathbb{P}_{26} = \emptyset$ and…)

c. δ is an edge, $K_\delta \cap O = \{\sigma, \tau\}$, where σ is 18-adjacent to τ and:

1. for $\overline{k} = 6$, and $k \neq 6$, $f(O, \alpha_1) = f(O, \alpha_2)$, where α_1, α_2 are the vertices of δ.
2. for $k, \overline{k} \in \{18,26\}$, σ and τ belong to distinct 6-components of $(K_{\alpha_1} \cup K_{\alpha_2}) \cap O$.





Separation properties

As consequences of the Jordan-Brower Theorem in continuous topology…

Th.: Each k-connected surface S in the universal $(k,\overline{k})$-space is a $\overline{k}$-separating set (separates its background into two $\overline{k}$-components); however it may be not strong separating.

Th. 15: [Relative Balls] Let S be a k-connected surface in the universal $(k,\overline{k})$-space. Then, for each vertex $c(\alpha) \in \mathcal{A}_S$, the difference

$$D_\delta = |\mathrm{st}(c(\alpha), \mathcal{A}_{R^3})| - |\mathrm{st}(c(\alpha), \mathcal{A}_S)|,$$

has exactly two components that characterize the $\overline{k}$-components of $\mathbb{Z}^3 - S$.

[Recall: $\mathrm{st}(c(\alpha), X) = \{A \in \mathcal{A}_X | \exists B \in \mathcal{A}_X$ such that $c(\alpha) < B$ and $A < B\}$]

Prop. 17: Let $\sigma \in S$ be a voxel, then the set $\cup_{c(\delta)\in X} K_\delta - S$ is contained in a $\overline{k}$-component of $N_{26}(\sigma) - S$ for each component X of D_σ.

Main Result

Th. 19: For $(k,\overline{k}) \neq (6,6)$, a subset $S \subseteq \mathbb{Z}^3$ is a k-connected simple $(k,\overline{k})$-surface if and only if it is a strong $\overline{k}$-separating k-connected surface in the universal $(k,\overline{k})$-space.

Proof ("if" part):

- each voxel $\sigma \in S$ is simple $(k,\overline{k})$-surface point:
  - S is $\overline{k}$-thin at σ

    Th. 18: A k-connected surface S in the universal $(k,\overline{k})$-space is strong $\overline{k}$-separating if and only if it is $\overline{k}$-thin.
  - σ is $\overline{k}_t$-linked to each voxel $\tau \in S$ which is strictly t-adjacent to it. (Prop. 21)
- the extra condition on unit cubes

  Prop. 20: If $K - S$ is not $\overline{k}$-connected, for a unit cube K, then $K - S \in \mathbb{P}_{\overline{k}}$ and it meets A_σ and B_σ for each voxel $\sigma \in K \cap S$.

Proof ("only if" part): rather long analysis of cases to check that the link $\mathrm{lk}(c(\alpha), \mathcal{A}_S)$ of each vertex $c(\alpha) \in \mathcal{A}_S$ is a 1-sphere (i.e., $\mathcal{A}_S$ is a combinatorial surface in the universal $(k,\overline{k})$-space).

Conclusions and further remarks

- We have generalized the notion of simple surface point introduced by Morgenthaler & Rosenfeld in 1981, that yields a new definition of simple surface.
- For $(k,\overline{k}) \neq (6,6)$, simple surfaces are the strong separating digital surfaces of the universal $(k,\overline{k})$-space, and thus
  - strong and simplicity surfaces are examples of simple surfaces
  - up to 10,580 different configurations are possible in the 26-neighbourhood of a voxel.
- What about simple (6,6)-surfaces?

  To prove directly that a simple surface is a strong separating)
- Is it possible to drop the extra condition on non connected unit cubes from the definition of a simple surface?

  Change the notion of simple surface point:

  Two voxels $\sigma, \tau \in O$ are $\overline{k}_6$-linked to each other if $A_\sigma \cap A_\tau \neq \emptyset \neq B_\sigma \cap B_\tau$ while $A_\sigma \cap B_\tau = \emptyset = B_\sigma \cap A_\tau$.